

Term Structure of Discount Rates under Multivariate $\vec{\mathfrak{S}}$ -Ordered Consumption Growth

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Abstract

The statistical relationship among future changes in consumption can be used to derive, under certain assumptions on investor preferences, an unambiguous effect on the term structure of discount rates. Thus, an increase in concordance in uncertain consumption growth has a negative impact on the term structure if, and only if, the representative investor is risk-averse and prudent (Gollier, *Pricing the Planet's Future*, Princeton University Press, 2013). Using multivariate $\vec{\mathfrak{S}}$ -concave stochastic orderings, this paper generalizes this relationship to multivariate higher-order risk preferences. The result under concordance is included for bivariate (1,1)-increasing concave orders. Similar generalizations arise for the good-specific discount rates and their relationships in a stochastic multi-good economy. In an approximate representation of the interest rate for the univariate case, the term-structure effects are controlled by the Ross coefficients of risk aversion. The effect on the term structure decreases with initial consumption for a given stochastic deterioration in the future consumption increments.

Keywords: term structure of discount rates, multivariate stochastic orders, multivariate higher-order risk aversion, good-specific discount rates, precautionary effect

JEL classification: H43, E43, D81

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1 Introduction

This paper explores a recent multivariate generalization in the theory of integral stochastic orderings for the theory of the term structure of discount rates.¹ The stochastic orders referred to state the equivalence of the stochastic dominance relationship between two random vectors (with equal support, bounded from below) and the ordering of the expectations of functions from certain classes which have either of the two random vectors as their argument. I focus on orders of the \vec{s} -concave type. The vector $\vec{s} \equiv (s_1, \dots, s_n)$ of positive integers s_i , $i = 1, \dots, n$, collects the stochastic dominance orders between the corresponding components of the two random vectors. The term concave refers to the sign of the highest (partial) derivative of the functions involved which alternates among each pair of adjacent (partial) derivatives, starting with a positive very first. In economic or actuarial contexts, these functions are typically interpreted as utility functions. As a consequence, their arguments constitute vectors of quantities of goods, and the sign conditions on the derivatives of the functions have an interpretation in terms of higher-order (multivariate) risk attitudes. The stochastic orders allow thus for a partial ordering of probability distributions over outcomes based on only partial information of the decision maker's utility.

Two kinds of previous results motivate the present analysis. First, a representative investor's risk-averse and prudent attitude is necessary and sufficient for a negative effect on the term structure of the efficient discount rate from concordance increases between two future random increments to initial consumption.² The intuition is that due to the increased concordance in future consumption a negative or positive effect on next period's consumption propagates more strongly on subsequent consumption, implying a mean-preserving spread of future consumption. A prudent individual will dislike this effect and thus reinforce precautionary behavior. As a consequence, future consumption will receive a higher weight for the present. Second, Gollier (2010, 2013) describes cross-effects between good-specific discount rates relating, respectively, to the complementarity (substitution effect), mutual riskiness (cross-prudence effect), and correlation of the riskiness of the two goods (correlation effect). Necessary and sufficient for these cross-effects are specific bivariate preferences such as correlation aversion (for the substitution effect) and cross-prudence.³

¹The theory of univariate s -convex integral stochastic orderings for real-valued measurable functions was developed by Denuit et al. (1998), following earlier contributions by Whitt (1986), Marshall (1991), and Müller (1997a,b), and extended to the bivariate case by Denuit et al. (1999a,b). Denuit and Eeckhoudt (2010a) treat the bivariate concave case with some economic applications. This paper relies on the multivariate cases of Denuit and Mesfioui (2010) and Denuit et al. (2010b), the latter treating the concave counterpart. The concepts referred to are introduced in detail in Section 2.

²The announced statement is a variant of Gollier's (2007, 2012, 2013) proposition under concordance (*cf.* Subsection 4.1). Concordance describes the intensity of association between the variables in the domain of a function at comparable levels of their identical discrete univariate domains (*e.g.*, Epstein and Tanny 1980, Tchen 1980).

³A correlation averse individual prefers a lottery with equally probable mixed good and bad outcomes to a lottery that yields either good or bad outcomes with equal probabilities. Cross-prudence describes the preference to temper a risk on one outcome by a sure increase in the other outcome (*cf.* Section 2 below).

These two kinds of results may be associated with two different approaches to determine the present value of a project which generates multidimensional impacts over a long time horizon. The first refers to the standard approach to first determine the certainty equivalent of future net benefits at the time of their occurrence and then to discount them at a single and unique rate to the present. The second approach, first proposed by Malinvaud (1953), is to discount every good at its specific rate, taking into account the specific uncertainty evolution and relative scarcities related to it, and then to aggregate the present values. The two strategies are strictly equivalent if the evolution of uncertainty, relative scarcities and substitutabilities as well as future preferences are perfectly known for every good. However, especially if applied under knowledge restrictions, the multivariate approach may lead to changes in emphasis and possibly different decisions as compared to the standard approach. In view of the many decision problems with multivariate stochastic intertemporal trade-offs, it seems desirable to have a framework for single-good and good-specific discounting allowing to derive results that encompass an arbitrary number of periods or goods and stochastic-dominance relations of arbitrary order in each component.

Using the recent multivariate generalizations of integral stochastic orderings, this paper provides a formalism and results that state the effects on single-good discount rates and cross-effects on other goods' discount rates from stochastic deteriorations of arbitrary order among an arbitrary number $n \geq 1$ of increments to initial consumption given higher-order univariate or multivariate risk aversion. These multivariate integral orders have the advantages of covering many of the stochastic orders used in economics as particular cases, including, for example, stochastic dominance, the correlation order and the concordance order (*e.g.*, Denuit and Mesfioui 2010). Their formulation involves a fairly tractable formalism. Moreover, their definition has an immediate link to higher-order (multivariate) risk preferences. The central vehicle for the proofs in this paper is a lemma stating that, for appropriate risk preferences, the expected marginal utility premium of a stochastic deterioration in the \vec{s} -concave order is non-negative.

The main result, in Section 3, has three parts. First, any stochastic deterioration in future consumption, whose increments in n future periods are related in a \vec{s} -concave order, has a negative effect on the efficient (single-good) discount rate if and only if the representative agent is $(\sum_{i=1}^n s_i + 1)^{th}$ -degree risk averse, in the sense of Ekern (1980). This extended precautionary effect immediately extends the well-known precautionary effect on the social discount rate to n future consumption periods and higher-order effects. Second, following the approach of good-specific discount rates, the described stochastic deterioration in the \vec{s} -concave order in the consumption increments of one good will have a negative effect on the term structure of another good's specific discount rate if and only

if the representative agent is cross- $(\sum_{i=1}^n s_i, 1)$ risk averse. This is an immediate generalization of the cross-prudence effect. Third, a stochastic deterioration in the distribution of future consumption levels of all goods has a negative effect on a good-specific discount rate if and only if the representative investor exhibits multivariate risk aversion and is open to a cross-effect with respect to that good. Multivariate- \vec{s} risk aversion is related below to the non-negative (non-positive) $(\sum_{i=1}^n s_i)^{th}$ partial derivative of the agent's utility function for $\sum_{i=1}^n s_i$ odd (even). The third part is the multivariate higher-order generalization of Gollier's correlation effect.

Various applications of the good-specific framework beyond the bivariate case are conceivable. For example, the differential discounting of economic and environment-improving investments in Gollier (2010, 2013) might be amended by health. For each attribute, different ways to influence their future development can be conceived. Investing in a project that uses a known technology may yield an almost deterministic return in one or more attributes, while investing in a technological research and development (R&D) project may impact on the evolution of each of the attributes in a different way and shape their stochastic relations. As another example, consider the social decision to combat low employment rates among the young by investing in university education *versus* a program to expand unskilled employment. Studying is likely to imply less consumption and less leisure in the short run, but an at least as high consumption and enjoyment of leisure in the future as well as, in the end, a longer life. As a third case with multidimensional stochastic intertemporal impacts, in health, the environment, consumption, and leisure, one might think of attempts to reduce pesticides in agriculture.

Section 4 collects some further results for single-good discount rates in the univariate case. The cited result under concordance is shown to correspond to a stochastic deterioration in the (1,1)-increasing concave order under a risk-averse and prudent representative investor. In the case of univariate risk preferences, the effects on the economy's equilibrium interest rate can be separated in the summands of a Taylor approximation of it, each of which is controlled by the corresponding Ross coefficient of $(k + 1)^{th}$ -degree risk aversion (as defined by Denuit and Eeckhoudt 2010b). Moreover, the strength of the impact on the yield curve of a deterioration of the random addends to initial consumption in the \vec{s} -concave order decreases with rising initial consumption. This result extends similar findings by Eeckhoudt et al. (2009) and Denuit and Rey (2010), who referred to an expected utility premium, to the case of an expected *marginal* utility premium.

This paper stands in the long lineage of research in economics interested in multi-attribute settings, eventually in a risky context, and in related preferences also under interdependent attributes (*e.g.*, Malinvaud 1953, Stiglitz 1969, Keeney 1973, Kihlstrom and Mirman 1974, Levy and Paroush 1974, Richard 1975, Duncan 1977, Karni 1979, Epstein and Tanny 1980, Atkinson and Bourguignon 1982,

Jouini et al. 2013). Recently, a revived interest has concerned the conceptualization of the stochastic relationships between economic variables in multivariate contexts. Meyer and Strulovici (2012) analyze the relation of the five interdependence orders of greater weak association (or correlation order), and the supermodular, convex-modular, dispersion, and concordance orders. They find that, while all five orders are equivalent in the bivariate case, for three or more dimensions one strictly implies the next in the given order, with the exception that for three dimensions the last two are equivalent. In a multivariate generalization of Rothschild and Stiglitz (1970) mean-preserving spreads, Müller and Scarsini (2012) show that the dominating of two probability measures that are ordered in the inframodular order can be obtained by a finite number of inframodular transfers. These transfers can, under suitable conditions, be decomposed into two more basic transformations, one decreasing correlation without altering the marginal distributions of the attributes, the other reducing dispersion in one attribute without affecting the correlation between the attributes. Decancq (2012) provides a characterization of the multivariate concordance order combining a representation of multivariate first-order stochastic dominance on a Fréchet class in terms of elementary rearrangements based on the joint distribution function and an alternative geometric generalization of bivariate elementary rearrangements for the corresponding survival function. With a more applied interest, Gravel and Moyes (2012), Le Breton et al. (2012), and Muller and Trannoy (2012) develop test criteria for more general multi-attribute rankings than have been available before.

2 Conceptual Basis

A stochastic ordering \preceq_* between two n -dimensional random vectors, $\vec{\mathbf{X}}, \vec{\mathbf{Y}}$, is typically defined as the ordering of the expectations of measurable functions u of some class $\mathcal{U}_*^{\mathcal{S}^n}$ defined over \mathcal{S}^n , with $\mathcal{S} \subseteq \mathbb{R}$, having either of the two random vectors as their argument,

$$\vec{\mathbf{X}} \preceq_* \vec{\mathbf{Y}} \quad \Leftrightarrow \quad Eu(\vec{\mathbf{X}}) \leq Eu(\vec{\mathbf{Y}}) \quad \forall u \in \mathcal{U}_*, \quad (1)$$

provided that the respective expectations exist.⁴ If $\vec{\mathbf{X}} \preceq_* \vec{\mathbf{Y}}$ holds, then $\vec{\mathbf{X}}$ is said to be smaller than $\vec{\mathbf{Y}}$ in the stochastic ordering \preceq_* generated by \mathcal{U}_* . Alternatively, $\vec{\mathbf{Y}}$ is said to dominate $\vec{\mathbf{X}}$ with respect to \mathcal{U}_* . The difference of the expected utilities $Eu(\vec{\mathbf{X}}) - Eu(\vec{\mathbf{Y}})$ will also be called *expected utility premium*, and the difference between the expected marginal utilities $E \frac{\partial u(\vec{\mathbf{X}})}{\partial X_i} - E \frac{\partial u(\vec{\mathbf{Y}})}{\partial Y_i}$, for some $i \in \{1, \dots, n\}$, *expected marginal utility premium*.⁵

⁴As set out in footnote 1, the definition of multivariate generalized integral orders in this section follows Denuit et al. (2010b) and Denuit and Mesfioui (2010).

⁵The concept of the utility premium was introduced for the univariate case by Friedman and Savage (1948) for the difference between the expected utility of a risky choice and the utility of the expectation of the risky argument. It was

The (multivariate) \vec{s} -increasing concave stochastic ordering $\preceq_{\vec{s}-icv}$ refers to the class of all differentiable \vec{s} -increasing concave functions $u : \mathcal{S}^n \rightarrow \mathbb{R}$, defined as

$$\mathcal{U}_{\vec{s}-icv}^{\mathcal{S}^n} := \left\{ u \left| (-1)^{\sum_{i=1}^n k_i} \frac{\partial^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n \partial x_i^{k_i}} u(\vec{x}) \leq 0 \text{ for } k_i = 0, 1, \dots, s_i, i = 1, \dots, n, \sum_{i=1}^n k_i \geq 1 \right. \right\}.$$

The term increasing relates to the non-negative sign of the first derivative, the term concave to the non-positive second derivative of the functions in these classes. The definition covers, for example, stochastic dominance of degree s for \mathcal{U}_{s-icv} with $\mathcal{S} = \mathbb{R}_+$. If the sign conditions on all derivatives but the highest in each component are suppressed, the \vec{s} -concave order $\preceq_{\vec{s}-cv}$ arises which refers to the class of all differentiable \vec{s} -concave functions $u : \mathcal{S}^n \rightarrow \mathbb{R}$,

$$\mathcal{U}_{\vec{s}-cv}^{\mathcal{S}^n} := \left\{ u \left| (-1)^{\sum_{i=1}^n s_i} \frac{\partial^{\sum_{i=1}^n s_i}}{\prod_{i=1}^n \partial x_i^{s_i}} u(\vec{x}) \leq 0 \right. \right\}.$$

Close relatives of these functional classes are the corresponding classes of increasing-convex and convex functions where, respectively, all derivatives and all highest derivatives are non-negative. For the mixed partial derivative of a function u up to k_i in component i for $i = 1, \dots, n$ below also the notation $u^{(\vec{k})}$ will be used.

For the relationship between multivariate orders and univariate stochastic dominance between positive linear combinations of their respective components, the following relation holds.⁶

Remark 1

Let \vec{X}, \vec{Y} be two n -dimensional vectors with components X_i, Y_i for $i = 1, \dots, n$. It holds that

$$\vec{X} \preceq_{\vec{s}-cv} \vec{Y} \Rightarrow \sum_{i=1}^n \alpha_i X_i \preceq_{(\sum_{i=1}^n s_i)-cv} \sum_{i=1}^n \alpha_i Y_i \text{ for all } \alpha_i \geq 0, i = 1, \dots, n.$$

Proof. For the multivariate function $v(\vec{z}) = u(\sum_{i=1}^n \alpha_i z_i)$ with $u \in \mathcal{U}_{(\sum_{i=1}^n s_i)-cv}^{\mathcal{S}}$, it is $v^{(\vec{s})}(\vec{z}) = u^{(\sum_{i=1}^n s_i)}(\sum_{i=1}^n \alpha_i z_i) \prod_{i=1}^n \alpha_i^{s_i}$, so that $v \in \mathcal{U}_{\vec{s}-cv}^{\mathcal{S}^n}$. ■

In the following, I will interpret all functions u as utility functions. For components s_i of \vec{s} , $i = 1, \dots, n$, sufficiently high, the (mixed partial) derivatives contained in the definitions of the classes of functions have an interpretation in terms of higher-order uni- or multivariate risk aversion.⁷

later taken up, in particular, by Hanson and Menezes (1971) and Eeckhoudt and Schlesinger (2006, 2009). Denuit and Rey (2010) refer to an expected utility premium in the sense used here.

⁶Remark 1 holds analogously for increasing-concave functions. Denuit and Mesfioui (2010) prove the increasing-convex case also for general non-negative functions $\psi \in \mathcal{U}_{\vec{s}-icx}^{\mathcal{S}}$, where $\psi(\vec{x}) = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \geq 0$ for $i = 1, \dots, n$, is a special case. Eeckhoudt et al. (2009: Theorem 3) prove a version of the bivariate increasing concave case.

⁷The first part of Definition 1 refers to Ekern (1980), except for the formulation with weak inequality here. The second part formally extends Richard's (1975) multivariate risk aversion, contained for $\vec{s} = (2, \dots, 2)$, to higher orders.

Definition 1

An agent is s^{th} -degree risk averse if and only if $(-1)^s u^{(s)} \leq 0$ with $s > 1$. An agent is multivariate- \vec{s} risk averse if and only if $(-1)^{\sum_{i=1}^n s_i} u^{(\vec{s})} \leq 0$ with $s_i \geq 1$ for $i = 1, \dots, n$ and $\sum_{i=1}^n s_i > n$.

Definition 1 covers component-wise for every $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, for $s_i = 2$ the usual notion of risk aversion (in the sense of a concave utility function), for $s_i = 3$ prudence, for $s_i = 4$ temperance, for $s_i = 5$ edginess, and for $s_i \geq 6$ the corresponding higher orders of risk apportionment (*e.g.*, Eeckhoudt and Schlesinger 2006). Moreover, $u \in \mathcal{U}_{s\text{-}icu}$ with $s \rightarrow \infty$ corresponds to mixed risk aversion in the sense of Caballé and Pomansky (1996). For $n = 2$, bivariate- (s_1, s_2) risk aversion coincides for $(s_1, s_2) = (1, 1)$ with correlation aversion (as in Epstein and Tanny 1980), for $(s_1, s_2) = (2, 1)$ ($(s_1, s_2) = (1, 2)$) with cross-prudence in the first (second) outcome and for $(s_1, s_2) = (2, 2)$ with cross-temperance (as in Eeckhoudt et al. 2007).⁸

For the univariate case with $z \in \mathcal{S}$, it will be referred to $-u^{(n+1)}(z)/u^{(n)}(z)$ as the (*Arrow-Pratt*) coefficient of absolute n^{th} -degree risk aversion, to $-zu^{(n+1)}(z)/u^{(n)}(z)$ as the (*Arrow-Pratt*) coefficient of relative n^{th} -degree risk aversion, and to $-u^{(n+1)}(z)/u'(z)$ as the *Ross coefficient of (absolute) n^{th} -degree risk aversion*.⁹

The multivariate orders defined above require the support of the random variables X_i, Y_i on the real line to be bounded from below, for $i = 1, \dots, n$. Without loss of generality, I focus in the following on random variables X_i, Y_i with compact supports $[a_i, b_i]$, $a_i < b_i$, and further restrict the attention to $a_i = 0$ to facilitate notation. Note that all order relations considered in this paper are invariant under shifts and that, even though future consumption increments may well be negative, because they are added to initial consumption, the bounds of the related integrals will in effect be positive.

I assume, moreover, that the random vectors \vec{X} and \vec{Y} , with probability density functions $f_{\vec{X}}(\vec{x})$ and $f_{\vec{Y}}(\vec{y})$, respectively, have the same h -variate marginals for $h = 1, \dots, n - 1$,

$$\underbrace{\int \cdots \int}_{n-h} f_{\vec{X}}(\vec{t}) \prod_{\ell=1}^{n-h} dt_{i(\ell)} = \underbrace{\int \cdots \int}_{n-h} f_{\vec{Y}}(\vec{t}) \prod_{\ell=1}^{n-h} dt_{i(\ell)}, \quad (2)$$

where $i(\ell) \neq i(\ell')$ for $i \in \{1, \dots, n\}$, $\ell \neq \ell'$ and all $\ell \in \{1, \dots, n - 1\}$. This assumption makes the

⁸For cross-prudence, I follow Gollier's (2010, 2013) re-definition, inverting the arguments compared to Eeckhoudt et al. (2007).

⁹The absolute and relative coefficients of n^{th} -degree risk aversion that refer to successive derivatives of the univariate utility function have been introduced, respectively, by Caballé and Pomansky (1996) and Eeckhoudt and Schlesinger (2008). Ross (1981) proposes stronger criteria that also allow for comparisons of risk aversion with regard to reductions in risk, instead of its elimination. (Pratt (1990) notes some caveats to Ross' approach that apply to its later extensions, too.) Modica and Scarsini (2005) extend Ross' approach to downside risk aversion. Jindapon and Neilson (2007) develop an original approach to higher-order Arrow-Pratt and Ross risk aversion based on comparative statics. Denuit and Eeckhoudt (2010b) generalize the Ross/Modica-Scarsini approach to the n^{th} order, and also provide the link to s -increasing concave orders and Ekern increases in risk.

above defined orders of dependence. It will prove important in the proof of Lemma 1.

3 Main Result

Consider a multi-good economy with m goods in which at any date t the representative agent's felicity is a function, u , of the available quantities $\vec{c}_t = (c_t^1, c_t^2, \dots, c_t^m)$ contained in the class of all differentiable \vec{s} -concave functions $u : \mathcal{S}^m \rightarrow \mathbb{R}$. The kind of the goods is *a priori* not specified. For example, among four attributes the first could be an aggregate consumption good, the second leisure, the third an index of environmental quality, and the fourth health.¹⁰ The discount rate of good $j \in \{1, \dots, m\}$ can be derived by considering a simple marginal project that increases future consumption of that good by some sure amount at the expense of part of its current consumption leaving all other goods unaffected. The Euler condition for this problem is

$$\frac{\partial u(\vec{c}_0)}{\partial c_0^j} = e^{-\delta T} E \frac{\partial u(\vec{C}_T)}{\partial C_T^j} e^{r_T^j T}, \quad (3)$$

where δ is the individual rate of pure preference for the present, C_t^j is the consumption of good j at date t , which is only certain – and thus denoted c_0^j – at date 0, T is the project's time horizon, and r_T^j is the internal per-period rate of return of the described project at date 0.¹¹ Thus, equation (3) equates the welfare cost of reducing consumption of that good by one monetary unit, which is invested in the project, and the welfare benefit that such an investment yields. An investment of one unit at date 0 will increase consumption of that good at date T by $e^{r_T^j T}$. Expected marginal utility at date T will amount to $E \frac{\partial u(\vec{C}_T)}{\partial C_T^j} e^{r_T^j T}$, which is discounted to the present at rate δ . The equilibrium interest rate of a marginal investment in good j in this economy derives then from equivalence transformations of equation (3) as

$$r_T^j = \delta - \frac{1}{T} \ln \left(\frac{E \frac{\partial u(\vec{C}_T)}{\partial C_T^j}}{\frac{\partial u(\vec{c}_0)}{\partial c_0^j}} \right). \quad (4)$$

For frictionless and efficient markets, r_T^j coincides with the socially efficient discount rate for good j for maturity T . Expecting positive consumption growth, so that $EC_T^j > c_0^j$, $\ln(\cdot)$ is negative. Hence, a rise (decrease) in the expected marginal utility of future consumption $E \frac{\partial u(\vec{C}_T)}{\partial C_T^j}$ will reduce (increase) the long-term risk-free rate r_T^j .

¹⁰The definition of the attributes should, of course, be consistent with the \vec{s} -concave nature of the utility function chosen here. Alternatively, the analysis could be extended to other types of utility functions.

¹¹Obviously, equation (3) is equivalent to $e^{-\delta T} \frac{E \frac{\partial u(\vec{C}_T)}{\partial C_T^j}}{\frac{\partial u(\vec{c}_0)}{\partial c_0^j}} e^{r_T^j T} = 1$, which identifies the stochastic discount factor for good j as $E \left[e^{-\delta T} \frac{\partial u(\vec{C}_T)/\partial C_T^j}{\partial u(\vec{c}_0)/\partial c_0^j} \right]$. If good j is a zero-coupon bond, the solution for the per-period rate of return (or yield) r_T^j (as in equation (4) below) yields the standard asset-pricing formula for risk-free bonds (*e.g.*, Cochrane 2005).

For the interpretation of the results below, the three effects determining the interest rate for a single good *ceteris paribus* in the absence of predictability shall be briefly restated. In addition to the positive impatience effect (as expressed by δ), two competing effects occur. Higher expected consumption in the future makes individuals save less, and thus consume more, in the present. This growth effect raises the equilibrium interest rate.¹² However, the accumulating uncertainty in the future makes individuals save more. This precautionary effect reduces the equilibrium interest rate. The total effect on the interest rate depends on the relative strength of the two latter effects.

For a two-good economy, Gollier (2010, 2013) derives, moreover, three cross-effects that operate on good-specific discount rates. The substitution effect captures the impact of deterministic growth in the consumption of one good *ceteris paribus* on the other good's specific discount rate. This impact is positive (negative) if and only if the marginal utility of one good decreases (increases) with the other good, so that the two goods are substitutes (complements).¹³ In particular, the case of substitutes coincides with supermodularity of marginal utility or correlation aversion. The cross-prudence effect represents the impact of an increase of the riskiness of the other good's future consumption level (such as adding a zero-mean risk when the consumption levels are *a priori* certain) on the first good's specific discount rate. Its impact is negative (positive) if and only if marginal utility of the one good is convex (concave) in the other good and, thus, the agent exhibits cross-prudence (cross-imprudence). Finally, concordance increases between the random growth rates of the two goods decrease (increase) the efficient good-specific discount rate of a good if and only if the agent is cross-prudent (cross-imprudent) in that good.

The precautionary effect and the cross-effects are now considered in the given multi-good economy. In order to investigate the effect of a shift in the stochastic relationship in future consumption on good j 's equilibrium interest rate r_T^j when the per-period consumption increments are related across time in a $\bar{\mathbf{s}}$ -concave order, the time horizon of the considered project is divided in n subperiods $(t_{i-1}, t_i]$, $i = 1, \dots, n$. Let $Z_i^j \in \{X_i^j, Y_i^j\}$ denote the random increment to the initial consumption of good j in subperiod i , with characteristics as defined in Section 2. Consumption at the termination of the project arises then as¹⁴

$$C_T^j = c_0^j + Z_1^j + \dots + Z_n^j . \quad (5)$$

The vector of terminal consumptions of all goods is also denoted $\vec{C}_T^{\mathbf{Z}}$, pointing to the $n \times m$ matrix

¹²In a single-good economy, the growth effect coincides with the standard wealth effect.

¹³*Cf.* also Traeger (2011) for a detailed account of the derivation and time development of good-specific discount rates in the bivariate case under certainty and limited substitutability.

¹⁴Alternatively, one could have defined \vec{Z}^j as the vector of the (conditional) changes in log consumption in the n subperiods. However, because the expectation of the log consumption is not the same as the log of the expectation of consumption, this alters the nature of the comparative-static exercise leading to a much more complicated condition on preferences involving several recursions.

$\mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$ collecting the random increments to initial consumption of all m goods of the economy for the n considered subperiods which are included in $\vec{\mathbf{C}}_T$.

The term-structure effects can be seen by comparing the long-term interest rates r_T^j for the two alternative vectors of random increments, $\vec{\mathbf{X}}^j$ and $\vec{\mathbf{Y}}^j$, or of terminal consumptions of all goods, $\vec{\mathbf{C}}_T^{\mathbf{X}}$ and $\vec{\mathbf{C}}_T^{\mathbf{Y}}$, respectively, related in a certain stochastic order. The following theorem states the main result.

Theorem 1

Let $\mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$ be the $n \times m$ matrix collecting the random increments to initial consumption of all m goods of the economy for the n considered subperiods, let $\vec{\mathbf{Z}}^j$ be the column associated with good j and let $\vec{\mathbf{C}}_T^{\mathbf{Z}}$ be the vector of terminal consumptions of all goods with elements C_T^j defined as in equation (5). Consider

- (a) a shift in the vector of random adds to initial consumption of good j from $\vec{\mathbf{Y}}^j$ to $\vec{\mathbf{X}}^j$, where $\vec{\mathbf{Y}}^j$ is larger than $\vec{\mathbf{X}}^j$ in the $\vec{\mathbf{s}}$ -concave order, the random distributions of all other goods $j' \in \{1, \dots, m\} \setminus \{j\}$ being unchanged;
 - (b) a shift in the vector of terminal consumptions of all goods from $\vec{\mathbf{C}}_T^{\mathbf{Y}}$ to $\vec{\mathbf{C}}_T^{\mathbf{X}}$, where $\vec{\mathbf{C}}_T^{\mathbf{Y}}$ is larger than $\vec{\mathbf{C}}_T^{\mathbf{X}}$ in the $\vec{\mathbf{s}}$ -concave order.
1. **(Extended precautionary effect)** Any type-(a) shift reduces the long-term risk-free rate of good j if and only if the representative agent is $\left(\sum_{i=1}^n s_i^j + 1\right)^{th}$ -degree risk averse.
 2. **(Cross- $\left(\sum_{i=1}^n s_i^j, 1\right)$ risk-aversion effect)** Any type-(a) shift reduces the long-term risk-free rate of good $k \in \{1, \dots, m\} \setminus \{j\}$ if and only if the representative agent is bivariate- $\left(\sum_{i=1}^n s_i^j, s^k\right)$ risk averse with $s^k = 1$.
 3. **(Generalized correlation effect)** Any type-(b) shift reduces the long-term risk-free rate of good j if and only if the representative agent is multivariate- $\left(\vec{\mathbf{s}} + (0, \dots, 0, k^j, 0, \dots, 0)\right)$ risk averse with $k^j = 1$.

The proof relies on the following lemma, which signs the expected marginal utility premium of a stochastic deterioration in the $\vec{\mathbf{s}}$ -concave order for appropriate risk preferences.¹⁵

Lemma 1

For any function $-\frac{\partial u(\vec{\mathbf{Z}})}{\partial \mathbf{Z}} \in \mathcal{U}_{\vec{\mathbf{s}}-cv}^{S^n}$ with $\mathbf{Z} \in \{X, Y\}$ being some element of random vector $\vec{\mathbf{Z}} \in \{\vec{\mathbf{X}}, \vec{\mathbf{Y}}\}$, and any pair of random vectors, $\vec{\mathbf{X}}$ and $\vec{\mathbf{Y}}$, where the first precedes the second in the $\vec{\mathbf{s}}$ -concave order

¹⁵The NSD equivalence (e.g., Eeckhoudt and Schlesinger 2008) is a special case of Lemma 1 focused on univariate N^{th} -order stochastic dominance, requiring s^{th} -degree risk aversion for $s = 2, \dots, N + 1$.

$(\vec{X} \preceq_{\vec{s}-cv} \vec{Y})$, it holds that

$$E \frac{\partial u(\vec{X})}{\partial X} - E \frac{\partial u(\vec{Y})}{\partial Y} \geq 0. \quad (6)$$

The proof of Lemma 1 uses a statement of \vec{s} -increasing concave orderings that includes explicit sign conditions on the integrated left tails of the distributions of the two random vectors \vec{X} and \vec{Y} . The following two characterizations provide the link between the definition of $\preceq_{\vec{s}-icv}$ as in (1) with $\mathcal{U}_* = \mathcal{U}_{\vec{s}-icv}^{S^n}$ and the sign conditions. Characterization 1 provides conditions characterizing the multivariate \vec{s} -increasing concave order via lower partial moments, without reference to utilities. The proof is given in Appendix A for completeness.¹⁶

Characterization 1

Let \vec{X} and \vec{Y} be random vectors with support contained in $[0, \vec{b}]$, with $\vec{b} \in \mathcal{S}^n \subseteq \mathbb{R}_+^n$, and denote $x_+ = \max\{x, 0\}$. Then, $\vec{X} \preceq_{\vec{s}-icv} \vec{Y}$ if and only if

$$E \left(\prod_{i=1}^n (t_i - X_i)_+^{k_i-1} \right) - E \left(\prod_{i=1}^n (t_i - Y_i)_+^{k_i-1} \right) \geq 0 \quad (7)$$

for all $t_i \in [0, b_i]$ if $k_i = s_i$, and $t_i = b_i$ if $k_i = 1, \dots, s_i - 1$, $i = 1, \dots, n$.

An alternative characterization can be stated involving integral conditions. For a random vector \vec{Z} with distribution function $F(\vec{Z})$, the n -variate integrated left tails of \vec{Z} derive, starting from $F_{\vec{Z}}^{[1, \dots, 1]} \equiv F_{\vec{Z}}$, for $k_1, \dots, k_n \geq 1$ and all $i = 1, \dots, n$ as

$$F_{\vec{Z}}^{[k_1, \dots, k_i+1, \dots, k_n]}(\vec{z}) = \int_0^{z_i} F_{\vec{Z}}^{[k_1, \dots, k_i, \dots, k_n]}(z_1, \dots, t_i, \dots, z_n) dt_i, \quad (8)$$

where the superscript vector in square brackets indicates the number of integrations with respect to the elements of \vec{z} . Note that for $n = 1$, equation (8) states the standard integral condition for univariate stochastic dominance. A problem of technical nature is that $F_{\vec{Z}}^{[\vec{k}]}$ may be infinite for $n > 1$. However, each side of equation (8) is finite if the other is. In particular, $F_{\vec{Z}}^{[\vec{k}]}$ is finite if $E \left(\prod_{i=1}^n Z_i^{k_i} \right)$ exists. This can be seen from the following representation, which derives using induction and Fubini's Theorem,¹⁷

$$F_{\vec{Z}}^{[k_1, \dots, k_n]}(\vec{t}) = \frac{E \left(\prod_{i=1}^n (t_i - Z_i)_+^{k_i-1} \right)}{\prod_{i=1}^n (k_i - 1)!}. \quad (9)$$

Inserting representation (9) into the conditions in Characterization 1 yields

¹⁶Denuit and Mesfioui (2010), and Denuit et al. (2010b) contain statements of Characterization 1 for the convex and concave cases, respectively. The authors state the proof only for necessity explicitly.

¹⁷Scarsini (1985) and O'Brien and Scarsini (1991), for example, provide explicit proofs of particular cases.

Characterization 2

Let \vec{X} and \vec{Y} be random vectors with support contained in $[0, \vec{b}]$, with $\vec{b} \in \mathcal{S}^n \subseteq \mathbb{R}_+^n$. Then,

$$\vec{X} \preceq_{\vec{s}-icu} \vec{Y} \Leftrightarrow F_{\vec{X}}^{[k_1, \dots, k_n]}(\vec{t}) - F_{\vec{Y}}^{[k_1, \dots, k_n]}(\vec{t}) \geq 0$$

for all $t_i \in [0, b_i]$ if $k_i = s_i$, and $t_i = b_i$ if $k_i = 1, \dots, s_i - 1$, $i = 1, \dots, n$.

Using Characterization 2, Lemma 1 can now be proven.

Proof of Lemma 1. The sign conditions in Lemma 1 derive by induction. Let $D_{(\vec{s})}^n$ denote the difference of the expectations in conditions (6) for the n -dimensional random vectors \vec{X}, \vec{Y} . Consider the $(\sum_{i=1}^n s_i - n)$ -fold integration by parts of $D_{(\vec{s})}^n$, first for $n = 1$ and $s_1 \in \{1, 2\}$.

$$D_{(1)}^1 = \int_0^b u^{(1)}(t) d[F_X(t) - F_Y(t)] \geq 0 \quad (10a)$$

$$D_{(2)}^1 = - \int_0^b u^{(2)}(t) d[F_X^{[2]}(t) - F_Y^{[2]}(t)] \geq 0 \quad (10b)$$

The sign of $D_{(1)}^1$ arises because $u^{(1)} \geq 0$ and because the difference of the integrated left tails is non-negative, the sign of $D_{(2)}^1$ arises because $u^{(2)} \leq 0$ and because the difference of the integrated left tails is non-negative (*cf.* Characterization 2). Moreover, in line (10b) the definition of distribution functions implies that $\int d(F_X - F_Y) = 0$.

For a generic odd n , $s_1 \in \{1, 2\}$, and $s_j = 1$ for $j = 2, \dots, n$, where the role of s_1 among the s_i , $i = 1, \dots, n$, is chosen without loss of generality, it holds with analogous arguments that

$$D_{(1, \dots, 1)}^n = \int_0^{b_n} \dots \int_0^{b_1} u^{(1, \dots, 1)}(\vec{t}) d[F_{\vec{X}}(\vec{t}) - F_{\vec{Y}}(\vec{t})] \geq 0 \quad (10c)$$

$$D_{(2, 1, \dots, 1)}^n = - \int_0^{b_n} \dots \int_0^{b_1} u^{(2, 1, \dots, 1)}(\vec{t}) d[F_{\vec{X}}^{[2, 1, \dots, 1]}(\vec{t}) - F_{\vec{Y}}^{[2, 1, \dots, 1]}(\vec{t})] \geq 0, \quad (10d)$$

and for general \vec{s} with $\sum_{i=1}^n s_i$ even

$$D_{(\vec{s})}^n = - \int_0^{b_n} \dots \int_0^{b_1} u^{(\vec{s})}(\vec{t}) d[F_{\vec{X}}^{[\vec{s}]}(\vec{t}) - F_{\vec{Y}}^{[\vec{s}]}(\vec{t})] \geq 0. \quad (10e)$$

Lines (10d) and (10e), moreover, use the assumption of equal h -dimensional marginals for $h = 1, \dots, n - 1$. Obviously, for $D_{(1, \dots, 1)}^{n+1}$ and $D_{(2, 1, \dots, 1)}^{n+1}$ the signs are the same. For $D_{(\vec{s})}^{n+1}$, with $\sum_{i=1}^n s_i$ even, the following condition arises:

$$D_{(\vec{s})}^{n+1} = \int_0^{b_{n+1}} \dots \int_0^{b_1} u^{(\vec{s})}(\vec{t}) d[F_{\vec{X}}^{[\vec{s}]}(\vec{t}) - F_{\vec{Y}}^{[\vec{s}]}(\vec{t})] \geq 0. \quad (10f)$$

For n and $\sum_{i=1}^n s_i$ odd, it holds that

$$D_{(\vec{s})}^n = \int_0^{b_n} \cdots \int_0^{b_1} u^{(\vec{s})}(\vec{t}) d \left[F_{\vec{X}}^{[\vec{s}]}(\vec{t}) - F_{\vec{Y}}^{[\vec{s}]}(\vec{t}) \right] \geq 0 \quad (10g)$$

$$D_{(\vec{s})}^{n+1} = - \int_0^{b_n} \cdots \int_0^{b_1} u^{(\vec{s})}(\vec{t}) d \left[F_{\vec{X}}^{[\vec{s}]}(\vec{t}) - F_{\vec{Y}}^{[\vec{s}]}(\vec{t}) \right] \geq 0. \quad (10h)$$

Finally, note that the conditions (10) do not make use of sign conditions for $k_i < s_i$, for $i = 1, \dots, n$, and only exploit the conditions with $\vec{k} = \vec{s}$ from Characterization 2. As a consequence, conditions (6) hold for any $u \in \mathcal{U}_{\vec{s}-cv}^{S^n}$. ■

Note that, by Remark 1, Lemma 1 also holds if $\frac{\partial u(\vec{Z})}{\partial Z}$ is stated in a univariate way (for example, with the argument as specified in equation (5)). Theorem 1 can then be proven.

Proof of Theorem 1. Parts 1 and 2 follow by applying Lemma 1, with Remark 1, to the marginal distribution of good $j \in \{1, \dots, m\}$. Part 3 follows immediately from a version of Lemma 1 with \vec{C}_T^X substituted for \vec{X} and \vec{C}_T^Y for \vec{Y} . ■

In this setting, apart from the growth effect for every good (corresponding to the wealth effect in the univariate case), the extended precautionary effect occurs for every good with reference to a stochastic deterioration in the marginal distribution of the good's consumption levels. The substitution and cross-prudence effects of Gollier (2010, 2013) generalize to the cross- $\left(\sum_{i=1}^n s_i^j, 1\right)$ risk-aversion effect that represents the cross-effect of a respective stochastic deterioration in the marginal distribution of good j on the specific discount rate of good k for every pair of goods in the economy (*i.e.*, for all $j, k \in \{1, \dots, m\}$ with $j \neq k$). While the correlation effect in Gollier studies the impact of the addition of correlation as compared to a situation without correlation, the present *generalized* correlation effect captures the effect of a stochastic deterioration in the distribution of terminal consumption levels of all goods on the specific discount rate of good j . Note that while the cross- $\left(\sum_{i=1}^n s_i^j, 1\right)$ risk-aversion effect requires at least bivariate utility, the generalized correlation effect exploits the full generality of the multivariate formulation of Lemma 1.

4 Further Results

4.1 Relationship to Result under Concordance

The original motivation in Gollier (2007, 2012, 2013) has been to study term-structure effects that arise from predictability in future consumption, conceptualized in a general way. Gollier represents shifts in stochastic dependence in future consumption by referring to marginals-preserving increases

in concordance (MPIC) between the consumption growth rates in two future periods.¹⁸ He shows that a MPIC has a negative effect on the efficient discount rate if and only if the representative investor's relative prudence is larger than unity. Denuit et al. (2010a: Proposition 2.1) rely on the Epstein and Tanny (1980) concept of elementary correlation-increasing transformations to establish that for specific bivariate lotteries such shifts in stochastic dependence between the two elements of random couples correspond to stochastic deteriorations in the bivariate $\preceq_{(s_1, s_2)-icv}$ order. Thus, their result allows to generalize conditions for term-structure effects that refer to concordance increases in future consumption, such as Gollier's, to the case of bivariate $\preceq_{(s_1, s_2)-icv}$ orders. Adopting the present framework with additive random increments to initial consumption, which is also the basic one in Gollier (2007), part 1 of Theorem 1 above covers the bivariate stochastic deteriorations treated in Denuit et al. (2010a) as particular cases. Specifically, MPIC as considered by Gollier correspond to stochastic deteriorations in the (1,1)-increasing concave order. As a consequence, the original Proposition 2 in Gollier (2007) derives as a special case of part 1 of Theorem 1 for a stochastic deterioration in the (1,1)-increasing concave order. Remark 2 notes this result.

Remark 2

A marginals-preserving increase in concordance from the couple (Y_1, Y_2) of random increments to initial consumption to the more concordant couple (X_1, X_2) , corresponding to a stochastic deterioration in the (1,1)-increasing concave order, reduces the long-term risk-free rate if and only if the representative agent is risk-averse and prudent.

Obviously, part 1 of Theorem 1 remains still stronger than Remark 2 and its possible generalization based on Denuit et al. (2010a) as it is open to treating an arbitrary number $n \geq 1$ of future consumption periods and only refers to concave and not to *increasing* concave orders, the latter involving sign conditions on all partial derivatives of the functions.

A similar remark applies regarding the correlation effect of Gollier (2010, 2013), which also refers to the concordance concept and is generalized by part 3 of Theorem 1.

4.2 Illustration of Extended Precautionary Effect

For the univariate case, the effects on the discount rate and various characteristics of them can be illustrated at a simple approximate representation of the equilibrium interest rate r_T in equation (4). This representation arises exploiting a $(\sum_{i=1}^n s_i)^{th}$ -order Taylor approximation of $Eu'(C_T)$ and

¹⁸A MPIC between two discrete random variables Z_i , $i = 1, 2$, with $Z_i \in \{X_i, Y_i\}$ and $(Y_1, Y_2) \geq (X_1, X_2)$ without loss of generality, is defined as any transformation of their joint distribution such that $F_{(Y_1, Y_2)}$ is obtained from $F_{(X_1, X_2)}$ by shifting probability mass ε from a small neighborhood of the points (X_1, Y_2) and (Y_1, X_2) to a small neighborhood of (X_1, X_2) and (Y_1, Y_2) . Such an increase in concordance raises the correlation among the two random variables but does not affect their marginal distributions.

first-order Taylor approximations of $\ln Eu'(C_T)$ and $Eu'(C_T)$ as

$$r_T \approx \delta - \frac{1}{T} E(C_T - c_0) \cdot \frac{u''(c_0)}{u'(c_0)} - \frac{1}{T} \sum_{k=2}^{\sum_{i=1}^n s_i} \frac{E(C_T - c_0)^k}{k!} \cdot \frac{u^{(k+1)}(c_0)}{u'(c_0)} \quad (11)$$

If the random addends to present consumption c_0 in the n subperiods are independent and identically distributed, only the first three terms on the right-hand side of approximation (11) are relevant. Then, the strength of the (positive) wealth effect (second term) is controlled by the coefficient of absolute risk aversion, while the strength of the (negative) precautionary effect (third term) depends on the Ross coefficient of prudence.¹⁹

Consider now a stochastic deterioration among the n increments to present consumption in the \vec{s} -concave order. By Remark 1, Ekern (1980), and the assumption of equal marginals (*cf.* equation (2)), this deterioration is equal to an increase in $(\sum_{i=1}^n s_i)^{th}$ -degree risk in the sense of Ekern, where the first $(\sum_{i=1}^n s_i) - 1$ moments remain unaffected by the deterioration (due to the assumption of equal marginals) and the $(\sum_{i=1}^n s_i)^{th}$ moment increases (decreases) for $\sum_{i=1}^n s_i$ even (odd).²⁰ Assuming non-negative growth, all moments are non-negative. Note now that the Ross coefficients of risk aversion $-\frac{u^{(k+1)}(c_0)}{u'(c_0)}$ alternate in sign with increasing degree, being non-negative (non-positive) for $k + 1$ even (odd). As a consequence, equation (11) provides an immediate illustration of the extended precautionary effect in part 1 of Theorem 1, predicting a uniformly non-positive effect on the term structure. The strength of the negative impact on the discount rate due to a stochastic deterioration among the consumption increments in the \vec{s} -concave order is controlled by the Ross coefficient of $(\sum_{i=1}^n s_i + 1)^{th}$ -order risk aversion.

4.3 Effects under Varying Initial Consumption

How does the strength of the effect on the equilibrium discount rate alter under a given stochastic deterioration between the vectors of random addends when the initial consumption varies? The following remark studies this question at the expected marginal utility premium associated with the stochastic deterioration.

Remark 3

Consider Lemma 1 for the case with univariate argument, i.e., $\frac{\partial u(\vec{z})}{\partial z} = \frac{\partial u(\omega + \sum_{i=1}^n z_i)}{\partial z}$, where $\omega > 0$ is

¹⁹Gollier (2007) provides an alternative representation where the effects are controlled by the product of all (Arrow-Pratt) coefficients of relative risk aversion up to the respective degree.

²⁰The famous examples for Ekern increases in second-, third-, and fourth-degree risk in the literature are, respectively, mean-preserving spreads (Rothschild and Stiglitz 1970), increases in downside risk (Menezes et al. 1980), and increases in outer risk (Menezes and Wang 2005).

an arbitrary constant. Then, $u \in \mathcal{U}_{(\sum_{i=1}^n s_i) - cv}^S \cap \mathcal{U}_{(\sum_{i=1}^n s_i + 1) - cv}^S$ is equivalent to

$$\frac{d}{d\omega} \left[E \frac{\partial u(\omega + \sum_{i=1}^n X_i)}{\partial X} - E \frac{\partial u(\omega + \sum_{i=1}^n Y_i)}{\partial Y} \right] \leq 0 \quad \forall \omega \quad \text{and} \quad \sum_{i=1}^n s_i \geq 1.$$

Proof. In the case with univariate argument in the analogs of equations (10) in the proof of Lemma 1 only the integrand $\frac{\partial u(\omega + \sum_{i=1}^n t_i)}{\partial t}$ depends on ω . The equivalence follows. ■

With the substitution $\omega = c_0$, Remark 3 means that the effect of a stochastic deterioration among the random increments to initial consumption in the \vec{s} -concave order on the expected marginal utility of final consumption tends to decrease in strength as the level of initial consumption increases if and only if the representative agent is $(\sum_{i=1}^n s_i)^{th}$ and $(\sum_{i=1}^n s_i + 1)^{th}$ -degree risk averse. Obviously, the conditions on the preferences in Remark 3 are slightly more restrictive than in Theorem 1.

Remark 3 parallels some findings in the recent literature. Eeckhoudt et al. (2009: Theorem 3) show that bivariate stochastic dominance $(X_1, X_2) \preceq_{(s_1, s_2) - icv} (Y_1, Y_2)$ implies a preference for “disaggregating harms” in the form of $(Y_1 + Y_2, X_1 + X_2) \preceq_{(s_1 + s_2) - icv} (X_1 + Y_2, Y_1 + X_2)$.²¹ Their proof refers to the expected-utility premium $g(w) \equiv E[u(w + X)] - E[u(w + Y)]$, showing that, for $X \preceq_{s - icv} Y$ and $u \in \mathcal{U}_{(s+t) - icv}^S$, $g(w) \in \mathcal{U}_{t - icv}^S$. Denuit and Rey (2010) show that the pain, measured by $-g(w)$, from a deterioration of the random addends to initial wealth w in the s -increasing concave order decreases with rising initial wealth if the agent’s utility $u \in \mathcal{U}_{(s+1) - icv}^S$. Moreover, the expected-utility pain from a mean-preserving increase in correlation decreases with rising initial wealth if the agent’s utility $u \in \mathcal{U}_{(s_1 + s_2 + 1) - icv}^{S^2 \times [-p_1 p_2, p_1(1-p_2)]}$ (so that the bivariate utility of initial wealth and the correlation parameter ρ , $U(w, \rho)$, is supermodular). The result in Remark 3, obtained for somewhat more general preferences, relates to these findings in that it refers to an expected marginal utility premium.

5 Conclusion

This paper shows that stochastic deteriorations of any order in future consumption of a good tend to decrease the socially efficient discount rate of that good, given appropriate higher-order risk preferences of the representative investor. In a framework with good-specific discount rates, such stochastic deteriorations, moreover, have a negative cross-effect on the discount rate of any substitute, given appropriate bivariate higher-order risk preferences. Finally, for appropriate multivariate higher-order risk aversion a stochastic deterioration in the distribution of the future consumption levels of all goods tends to decrease any single good-specific discount rate if the representative investor is open to a cross-

²¹The authors state their result referring to 50–50 lotteries of the indicated outcomes, and using the concept of risk apportionment of Eeckhoudt and Schlesinger (2006) that can express risk preferences of any order without reference to any particular utility model.

effect with respect to that good. The ‘appropriate’ (multivariate) risk preferences in these results are the ones where the sign of the highest (partial) derivative of the utility functions involved alternates among each pair of adjacent (partial) derivatives, starting with a positive very first. The results derive for a multi-good economy with an arbitrary number of future consumption periods and stochastic deteriorations of arbitrary order. I conceptualize the stochastic relations in future consumption using integral stochastic orderings of the multivariate \bar{s} -concave type, as recently developed by Denuit and Mesfioui (2010) and Denuit et al. (2010b).

The analysis has an immediate link to studies on precautionary saving under risk changes. Lemma 1 is a multivariate generalization of the N^{th} -order stochastic dominance (NSD) equivalence stated in Eeckhoudt and Schlesinger (2008), which is central to their proofs of conditions for precautionary-saving increases. The latter paper for the univariate case and Denuit et al. (2011) and Li (2012), for example, for the bivariate case provide ample illustration for underlying precautionary behaviors and conditions that induce the effects studied here. These studies and applications could easily be extended to the multivariate case based on the present framework.

The conceptualization and empirics of stochastic consumption growth are part of an important debate on the social valuation of future net benefits from anthropogenic climate change (*e.g.*, Gollier 2013, Weitzman 2012). Multivariate integral stochastic orderings, being dependence orders under equal multivariate marginals, may have their role in such debates as a more general framework to conceptualize and analyze predictability than usual in current macroeconomics and finance.

The involved higher-order multivariate risk preferences are empirically still few understood. For the univariate case, Deck and Schlesinger (2010), Ebert and Wiesen (2011, 2013) and Noussair et al. (2013) conduct experiments that find prudence and temperance prevailing either in student samples or, for the last authors, a representative sample of the Dutch population. Even fewer research has concerned correlation aversion and related higher-order concepts. A first incentivized-choice experiment on intertemporal correlation aversion, by Andersen et al. (2012), confirms its prevalence for a representative sample of the Danish population.

Appendix

A Proof of Characterization 1

Necessity. $Eu(\vec{\mathbf{Z}})$, with $u \in \mathcal{U}_{\vec{\mathbf{s}}-icv}^{\mathcal{S}^n}$, $Z_i \in [0, b_i]$, $b_i > 0$, for $i = 1, \dots, n$, has, when viewed as a function of Z_1 around 0 for fixed z_2, \dots, z_n , an exact representation based on a Taylor series expansion and remainder as

$$Eu(\vec{\mathbf{Z}}) = \sum_{k_1=0}^{s_1-1} \frac{E\left(Z_1^{k_1}\right)}{k_1!} u^{(k_1,0,\dots,0)}(0, z_2, \dots, z_n) + \int_0^{b_1} \frac{E\left((t_1 - Z_1)_+^{s_1-1}\right)}{(s_1 - 1)!} u^{(s_1,0,\dots,0)}(t_1, z_2, \dots, z_n) dt_1 .$$

Applying, similarly, Taylor series expansions to $u^{(k_1,0,\dots,0)}(0, z_2, \dots, z_n)$ and $u^{(s_1,0,\dots,0)}(t_1, z_2, \dots, z_n)$, when viewed as functions of Z_2 around 0 for fixed z_3, \dots, z_n , yields

$$\begin{aligned} Eu(\vec{\mathbf{Z}}) &= \sum_{k_1=0}^{s_1-1} \sum_{k_2=0}^{s_2-1} \frac{E\left(Z_1^{k_1} Z_2^{k_2}\right)}{k_1! k_2!} u^{(k_1, k_2, 0, \dots, 0)}(0, 0, z_3, \dots, z_n) \\ &+ \sum_{k_1=0}^{s_1-1} \int_0^{b_2} \frac{E\left(Z_1^{k_1} (t_2 - Z_2)_+^{s_2-1}\right)}{k_1! (s_2 - 1)!} u^{(k_1, s_2, 0, \dots, 0)}(0, t_2, z_3, \dots, z_n) dt_2 \\ &+ \sum_{k_2=0}^{s_2-1} \int_0^{b_1} \frac{E\left((t_1 - Z_1)_+^{s_1-1} Z_2^{k_2}\right)}{(s_1 - 1)! k_2!} u^{(s_1, k_2, 0, \dots, 0)}(t_1, 0, z_3, \dots, z_n) dt_1 \\ &+ \int_0^{b_2} \int_0^{b_1} \frac{E\left((t_1 - Z_1)_+^{s_1-1} (t_2 - Z_2)_+^{s_2-1}\right)}{(s_1 - 1)! (s_2 - 1)!} u^{(s_1, s_2, 0, \dots, 0)}(t_1, t_2, z_3, \dots, z_n) dt_1 dt_2 . \end{aligned}$$

Repeating this proceeding component by component gives the general expansion formula:

$$\begin{aligned} Eu(\vec{\mathbf{Z}}) &= \sum_{\mathcal{S}} \sum_{i \in \mathcal{S}} \sum_{k_i=0}^{s_i-1} \int_0^{b_{j(n_{\overline{\mathcal{S}}})}} \dots \int_0^{b_{j(1)}} \frac{E\left(\prod_{i \in \mathcal{S}} Z_i^{k_i} \prod_{j \in \overline{\mathcal{S}}} (t_j - Z_j)_+^{s_j-1}\right)}{\prod_{i \in \mathcal{S}} k_i! \prod_{j \in \overline{\mathcal{S}}} (s_j - 1)!} \\ &\times \frac{\partial^{\sum_{i \in \mathcal{S}} k_i + \sum_{j \in \overline{\mathcal{S}}} s_j}}{\prod_{i \in \mathcal{S}} \partial x_i^{k_i} \prod_{j \in \overline{\mathcal{S}}} \partial x_j^{s_j}} u(\mathbf{t}_{\overline{\mathcal{S}}}) \prod_{j \in \overline{\mathcal{S}}} dt_j , \end{aligned}$$

where \mathcal{S} and $\overline{\mathcal{S}}$ form a partition of $\{1, 2, \dots, n\}$ (i.e., $\mathcal{S} \cup \overline{\mathcal{S}} = \{1, 2, \dots, n\}$ and $\mathcal{S} \cap \overline{\mathcal{S}} = \emptyset$), $n_{\overline{\mathcal{S}}} = \#\overline{\mathcal{S}}$, $\mathbf{t}_{\overline{\mathcal{S}}} \equiv \sum_{i \in \overline{\mathcal{S}}} t_i \mathbf{e}_i$, and $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$. Specifying the expansion of $Eu(\vec{\mathbf{Z}})$ for $\vec{\mathbf{Z}} \in \{\vec{\mathbf{X}}, \vec{\mathbf{Y}}\}$, and comparing $Eu(\vec{\mathbf{X}}) \leq Eu(\vec{\mathbf{Y}})$ as in definition (1) taking into account the sign conditions on the partial derivatives of $u \in \mathcal{U}_{\vec{\mathbf{s}}-icv}^{\mathcal{S}^n}$ shows the necessity of inequalities (7).

Sufficiency. The functions of the form $\prod_{i=1}^n (t_i - z_i)_+^{k_i-1}$ under the conditions of Characterization 1 constitute the minimal generators of $\mathcal{U}_{\vec{\mathbf{s}}-icv}^{\mathcal{S}^n}$ and are thus in this class of functions (Müller 1997a, Denuit and Mesfioui 2010). ■

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