Comparative Precautionary Saving under Higher-Order Risk and Recursive Utility

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Abstract

Measuring and comparing the precautionary saving motive rest almost exclusively on the expected utility framework, and only focus on income risk or coefficients of the Arrow-Pratt type. We generalize the standard approach by characterizing comparative precautionary saving under recursive utility for increases in income risk and increases in risk on the saving return, including higher-order risk effects. We express the comparisons in terms of precautionary premia. In addition, we define a new class of preference coefficients, and derive the associated conditions to predict a stronger precautionary motive. The coefficients provide a detailed picture of the preferences sustaining precautionary saving and could be useful in applications.

Keywords: precautionary saving, prudence, recursive utility, higher-order risk, precautionary premium, preference coefficient

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1 Introduction

Precautionary saving is a well-established feature of life-cycle consumption, essential to various fields including the design of social-security and tax systems (Mirrlees et al. 2010), social discounting (Gollier 2013), and asset allocation (Gomes and Michaelides 2005). Despite this ubiquity, its extent and drivers are still subject to debates (e.g., Gourinchas and Parker 2001, Attanasio and Weber 2010, Choi et al. 2017, Lugilde et al. 2017). An important impediment to better empirics could be conceptual restrictions. To date, measuring and comparing the intensity of precautionary saving motives rest almost exclusively on expected utility (EU) and only focus on income risk or coefficients of the Arrow-Pratt type. For more general preferences and alternative risky environments, preference measures that are theoretically founded and amenable to empirics are not available.

This paper extends characterizations of comparative precautionary saving to higher-order risk effects under recursive utility (RU) and return risk. We express the comparisons in terms of precautionary premia. In addition, we provide a new representation based on preference coefficients. The coefficients give a detailed picture of the different preferences sustaining precautionary saving and could prove useful in empirical applications.

Under EU, precautionary saving has been studied for a broad set of risks. Eeckhoudt and Schlesinger (2008) provide the conditions on the utility function guaranteeing higher saving for increases in income risk or return risk, including higher-order risk effects. Kimball's (1990) prudence coefficient and precautionary premium are only formulated for a risk added to nonrandom future income, and do not cover most of the risk events in Eeckhoudt and Schlesinger. Liu (2014) generalizes Kimball's precautionary-premium analysis to accommodate higher-order effects in income risk using an extension of comparative risk aversion in the sense of Ross (1981). Liu's contribution is a main ingredient in this paper.

Our analysis starts from the two-period consumption/saving model under RU in Kimball and Weil (2009) (KW). RU disentangles risk preferences and the elasticity of intertemporal substitution (EIS), and allows thus to account for risk and intertemporal tradeoffs not identified in the EU framework.[1] Under RU, risk impacts saving via two channels,

[1]For example, RU has helped to resolve the equity-premium and risk-free-rate puzzles in macroeco-

marginal expected utility (MEU) and the certainty equivalent (CE) of future consumption (Bostian and Heinzel 2018). In the MEU channel, precaution works as under EU, so that, for income risk, the contributions from this channel to the precautionary saving motive can be compared as in Liu. In the CE channel, the effect of a CE variation on saving depends on the simultaneous action of risk and intertemporal preferences. RU therefore helps to cover some less common intuitions. For example, the more risk-averse of two decision-makers, when having a large EIS, may hold the *smaller* precautionary motive.

Saving reactions to return risk follow a different intuition compared to income risk. Because risk exposure is directly endogenous to saving, a competing substitution effect adds to the precautionary effect. Thus, a prudent decision-maker in the Kimball sense – who saves more in response to income risk – could well save *less* when return risk increases. The multiplicative character of return risk implies that the premium and coefficient measures of the Arrow-Pratt type are not well defined in this context (Briys et al. 1989). To measure the precautionary saving motive under return risk, we adapt Eeckhoudt and Schlesinger's (2009) multiplicative risk premium. The comparative statements we derive resemble the case with income risk but involve an expression linked to the conditions for higher saving under EU. Our main finding is that the sign of the multiplicative precautionary premium follows the positive or negative direction of the saving reaction.

While this sign ambivalence is perfectly sound theoretically, it limits the practical use of these premia. Adding to the interpretative difficulty, the premia for multiplicative and additive risks have different units, and thus cannot be compared. Therefore, we define a new class of preference coefficients building on Liu and Meyer (2013a,b), and derive the associated conditions to predict a stronger precautionary motive. The coefficients for income risk also apply to return risk. But the separate coefficients, controlling the precautionary and substitution effects, make the conditions under return risk more involved.

Section 2 introduces the decision framework for the analysis. Sections 3 and 4 treat, in turn, the cases with income risk and return risk. Section 5 develops the representation of comparative precautionary saving using preference coefficients. Section 6 discusses how

nomics and finance (Hall 1988, Weil 1989, Epstein and Zin 1991, Tallarini 2000, Bansal and Yaron 2004, Barro 2009, Binsbergen et al. 2012, Martin 2013, Epstein et al. 2014).

our results fit within the existing literature. Section 7 concludes.

2 Decision Framework

Following KW, we consider a two-period consumption/saving model under RU. The decision-maker chooses the amount of saving s out of first-period income y_1 that maximizes the intertemporal utility objective

$$u\left(y_1 - s\right) + \beta u\left(CE(\tilde{y}_2 + s\tilde{R})\right) \tag{1}$$

Here, u is the felicity function capturing the preference for consumption smoothing, and β is the utility discount factor reflecting pure time preference. Second-period consumption $\tilde{c}_2$ consists of income $\tilde{y}_2$ and saving with gross return $\tilde{R}$. Risk may enter through either $\tilde{y}_2$ or $\tilde{R}$, and a tilde indicates that a variable is risky.[2] The certainty-equivalent function $CE(\tilde{c}_2)$ ranks future consumption according to the risk preference ψ:

$$CE(\tilde{y}_2 + s\tilde{R}) \equiv \psi^{-1}\left(E\left[\psi\left(\tilde{y}_2 + s\tilde{R}\right)\right]\right) \tag{2}$$

Unlike u, ψ is a von Neumann-Morgenstern utility function (Selden 1978). Special cases of (1) are EU (for $\psi = u$), and infinite-horizon Epstein and Zin (1989, 1991) and Weil (1990) utility (for constant relative risk aversion and constant EIS).

According to the first-order optimality condition

$$u'\left(c_1\right) = \beta u'\left(CE(\tilde{c}_2)\right) CE'(\tilde{c}_2) \tag{3}$$

the decision-maker saves until the marginal utility from foregoing consumption in period 1 (i.e., saving a marginal amount) is equal to the discounted marginal utility from consuming during period 2 instead. With some abuse of notation for compactness, CE' in (3) denotes

[2] If $\tilde{y}_2$ and $\tilde{R}$ simultaneously carry a tilde, the notation applies to each of the two cases.

the full derivative of the certainty equivalent with respect to *saving*:

$$CE'(\tilde{c}_2) \equiv \frac{dCE(\tilde{c}_2)}{ds} = \frac{E[\psi'(\tilde{c}_2)\tilde{R}]}{\psi'\left(CE(\tilde{c}_2)\right)} \tag{4}$$

Some restrictions must be placed on u and ψ to fulfill the second-order sufficiency condition. Namely, both must be increasing and concave, but ψ^{-1} must also be concave in s. This requirement is fulfilled if ψ's absolute risk tolerance – the inverse of absolute risk aversion – is concave (KW, Gollier 2001). Under these assumptions (which we maintain throughout), $CE''(\tilde{c}_2) < 0$, guaranteeing a negative second derivative:

$$u''\left(c_1\right) + \beta\left[u''\left(CE(\tilde{c}_2)\right)\left[CE'(\tilde{c}_2)\right]^2 + u'\left(CE(\tilde{c}_2)\right)CE''(\tilde{c}_2)\right] < 0$$

Under income risk, optimal saving may be negative, in which case the same return R applies as for positive saving. When considering return risk, we assume, similar to Eeckhoudt and Schlesinger (2008), that preferences and incentives are such that they imply positive saving choices, to simplify the exposition.

When analytically more convenient, we use, like KW, an alternative formulation of model (1) that applies the Kreps and Porteus (1978) operator $\phi(.) \equiv u(\psi^{-1}(.))$. The utility objective is then

$$u\left(y_1 - s\right) + \beta\phi(E\psi(\tilde{y}_2 + s\tilde{R}))$$

and the first-order condition reads

$$u'\left(c_1\right) = \beta\phi'(E\psi(\tilde{c}_2))E[\psi'(\tilde{c}_2)\tilde{R}] \tag{5}$$

(3) and (5) are equivalent if ψ is a continuous and monotonically-increasing function. For, if $E\psi(\tilde{c}_2)$ is well defined, then, also the certainty equivalent (2) is well defined.

ϕ's curvature is a key element in the analysis below. It depends on a simple relationship of second-order preferences: ϕ is convex (concave) if and only if the absolute resistance to intertemporal substitution, $ARIS_u$, is less (greater) than absolute risk aversion, ARA_ψ:

$$-\frac{u''\left(CE(\tilde{c}_2)\right)}{u'\left(CE(\tilde{c}_2)\right)} < (>) - \frac{\psi''\left(CE(\tilde{c}_2)\right)}{\psi'\left(CE(\tilde{c}_2)\right)} \tag{6}$$

Because relative resistance to intertemporal substitution is the inverse of EIS, this condition implies a direct comparison of intertemporal and risk preferences.[3]

The saving reaction to risk depends on shifts in future marginal utility. Rewriting it in (3) using (4) and (5) reveals that risk affects future marginal RU via two channels, the certainty equivalent $CE(\tilde{c}_2)$ and marginal EU $E[\psi'(\tilde{c}_2)\tilde{R}]$ (Bostian and Heinzel 2018):

$$u'\left(CE(\tilde{c}_2)\right)CE'(\tilde{c}_2) = \frac{u'\left(CE(\tilde{c}_2)\right)}{\psi'\left(CE(\tilde{c}_2)\right)}E[\psi'(\tilde{c}_2)\tilde{R}] = \phi'(E\psi(\tilde{c}_2))E[\psi'(\tilde{c}_2)\tilde{R}] \tag{7}$$

$E[\psi'(\tilde{c}_2)\tilde{R}]$ has a direct positive influence in (7), but the impact of CE is positive (negative) depending on whether ϕ is convex (concave). Comparing precautionary saving motives under RU involves thus comparing the preferences operating in each channel.

To compare risk attitudes, we adopt, like Liu, the notion of $(n/m)^{th}$-degree Ross more risk aversion, introduced by Liu and Meyer (2013b).

Definition 1 ($(n/m)^{th}$-Degree Ross More Risk Aversion) *For two utility functions $\psi_u(x), \psi_v(x)$ with $sgn\left[\psi_u^{(k)}(x)\right] = sgn\left[\psi_v^{(k)}(x)\right] = (-1)^{k+1}$ for $k = n, m$ and $n > m$, $\psi_u(x)$ is $(n/m)^{th}$-degree Ross more risk-averse than $\psi_v(x)$ if there exists a scalar $\lambda > 0$ such that*

$$\frac{\psi_u^{(n)}(x_a)}{\psi_v^{(n)}(x_a)} \geq \lambda \geq \frac{\psi_u^{(m)}(x_b)}{\psi_v^{(m)}(x_b)} \quad \textit{for all} \quad x_a, x_b \in [a,b] \subseteq \mathbb{R}_0^+ \tag{8}$$

Definition 1 covers the original Ross more risk aversion when $(n,m) = (2,1)$ (Ross 1981), and its n^{th}-degree extension when $(n,m) = (n,1)$ (Jindapon and Neilson 2007).

To apply this definition to RU, we take ψ_u and ψ_v to be the risk preferences associated with the intertemporal preferences u and v. For our purposes, the pairings (u,ψ_u) and (v,ψ_v) reflect RU preferences that are, like β_u and β_v, in general, different. For brevity, we usually drop the u and v subscripts when it is obvious that only one decision-maker's

[3]In frameworks with more than two periods, the convexity (concavity) of ϕ results in an increased propensity to have risk resolved as soon (late) as possible (Epstein et al. 2014).

preferences are under discussion.

We compare precautionary saving motives by considering well-defined increases in risk. We use a special case of n^{th}-degree stochastic dominance (nSD): Liu's n^{th}-degree first-ℓ-moments-preserving stochastic dominance (n-ℓ-MPSD) order.[4]

Definition 2 (n^{th}-degree ℓ-first-Moments-Preserving Stochastic Dominance) *For any integer ℓ with $1 \leq \ell \leq n-1$, $\tilde{x}_h$ is dominated by $\tilde{x}_l$ in the n-ℓ-MPSD order if $\tilde{x}_h \preceq_{nSD} \tilde{x}_l$ and $E\left(\tilde{x}_l^j\right) = E\left(\tilde{x}_h^j\right)$ for $j = 1, \ldots, \ell$.*

In Definition 2, n indicates the stochastic-dominance degree used to compare the "higher risk" lottery $\tilde{x}_h$ and the "lower risk" lottery $\tilde{x}_l$, and ℓ counts how many of their lower moments are identical. Assuming $\ell \geq 1$ excludes direct first-order effects. The n-ℓ-MPSD order has a flexible intermediate position between Ekern's (1980) increases in n^{th}-degree risk ($\ell = n-1$) and Denuit and Eeckhoudt's (2013) n^{th}-degree mean-preserving stochastic dominance ($\ell = 1$).[5]

3 Increases in Income Risk

Central to comparing the precautionary saving motives of RU decision-makers is the RU precautionary premium θ^{y_2}. Using the first-order conditions (3) and (5), we define this premium, equivalently, as the solution to[6]

$$u'(CE(\tilde{y}_{2,l} + sR - \theta_u^{y_2}))CE'(\tilde{y}_{2,l} + sR - \theta_u^{y_2}) = u'(CE(\tilde{y}_{2,h} + sR))CE'(\tilde{y}_{2,h} + sR) \quad (9a)$$

$$\phi_u'(E\psi(\tilde{y}_{2,l} + sR - \theta_u^{y_2}))E\psi'(\tilde{y}_{2,l} + sR - \theta_u^{y_2}) = \phi_u'(E\psi(\tilde{y}_{2,h} + sR))E\psi'(\tilde{y}_{2,h} + sR) \quad (9b)$$

$\theta_u^{y_2}$ is the safe reduction in $\tilde{y}_{2,l}$ that has the same impact on saving as increasing income risk from $\tilde{y}_{2,l}$ to $\tilde{y}_{2,h}$. Our prior assumptions ensure that the left-hand sides depend positively on $\theta_u^{y_2}$, and so $\theta_u^{y_2}$ rises with the level of future marginal RU.

[4]By $\preceq_\star$, we denote a stochastic-dominance relation, and $\star \in \{$nSD,n-ℓ-MPSD$\}$ specifies the order.

[5]To illustrate, mean-preserving spreads (Rothschild and Stiglitz 1970) are second-degree Ekern risk increases and 2-1-MPSD shifts; increases in downside risk (Menezes et al. 1980) are third-degree Ekern risk increases and 3-2-MPSD shifts; and increases in outer risk (Menezes and Wang 2005) are fourth-degree Ekern risk increases and 4-3-MPSD shifts.

[6]The risk and precautionary premia derive for saving at its optimal level in the high-risk state.

Within the CE channel, the relationship between CE and n-ℓ-MSPD deteriorations of $\tilde{y}_2$ is given by inserting the risk premium $\pi_\psi^{y_2}$ into the rewritten CE definition (2):

$$\psi(CE(\tilde{y}_{2,h} + sR)) = E\psi\left(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2}\right) = E\psi\left(\tilde{y}_{2,h} + sR\right) \tag{10}$$

Liu's Theorem 2 implies that the risk attitudes in (10) can equivalently be compared via $\pi_\psi^{y_2}$ and $(k/1)^{th}$-degree Ross more risk aversion for $k = \ell + 1, \ldots, n$.

Regarding the MEU channel, we know from Liu's Theorem 3 that the contributions to precautionary saving can equivalently be compared in terms of the von Neumann-Morgenstern (vNM) precautionary premium $\theta_\psi^{y_2}$ from

$$E\psi'(\tilde{y}_{2,l} + sR - \theta_\psi^{y_2}) = E\psi'(\tilde{y}_{2,h} + sR) \tag{11}$$

and $((k+1)/2)^{th}$-degree Ross more risk aversion for $k = \ell + 1, \ldots, n$. The left-hand side of (11) depends positively on $\theta_\psi^{y_2}$ due to risk aversion of ψ.

Inserting $\pi_\psi^{y_2}$ and $\theta_\psi^{y_2}$ into (9b) links the two channels to the total precautionary motive:[7]

$$\phi_u'(E\psi(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2}))E\psi'(\tilde{y}_{2,l} + sR - \theta_\psi^{y_2}) = \phi_u'(E\psi(\tilde{y}_{2,h} + sR))E\psi'(\tilde{y}_{2,h} + sR) \tag{12}$$

While $\theta_u^{y_2}$ always increases with $\theta_\psi^{y_2}$, $\pi_\psi^{y_2}$ has an ambiguous effect: $\theta_u^{y_2}$ increases (decreases) with $\pi_\psi^{y_2}$ if ϕ_u is concave (convex).

Interpersonal comparisons of precautionary saving motives in the face of large risks generally require to evaluate the two decision-makers' utilities initially at the same argument (KW, Liu).[8] That is, the two must be subject to the same incentives, and the low-risk state must be such that both choose identical saving amounts. As implied by the first-order conditions (3) and (5), the latter requirement can be more or less restrictive. For example, if both share the same discount factor, then, their utilities need to coincide

[7] KW derive a similar relation in their context referring to compensating premia. We focus instead on equivalent premia.

[8] Only trivial cases escape from this rule, such as comparisons with a risk-neutral decision-maker.

at the low-risk equilibrium to yield identical saving amounts at this point. If their discount factors differ, then, their utility functions must still be such that the same saving amounts are chosen at that point. In the following, we assume that this requirement is fulfilled for any two decision-makers who are compared.[9]

The characterization of comparative precautionary saving for two RU decision-makers who save the same amount under $\tilde{y}_{2,l}$ follows from (12). Observing $\theta^{y_2}_{\psi_u} \geq \theta^{y_2}_{\psi_v}$ provides an unambiguous signal: ψ_u has the stronger precautionary motive from the MEU channel. For the CE channel, the comparisons hinge on the curvatures of ϕ_u and ϕ_v. If one of them is concave and the other convex, the concave one unambiguously has the stronger precautionary motive. However, if both are concave, ϕ_u is associated with the stronger precautionary motive, if ϕ_u is more concave than ϕ_v and $\pi^{y_2}_{\psi_u} \geq \pi^{y_2}_{\psi_v}$. If both are convex, the reverse relations must hold.

Theorem 1 states the sufficient conditions for observing $\theta^{y_2}_u \geq \theta^{y_2}_v$ in the latter three cases that admit such conditions.[10] Because preferences operate in the two risk-impact channels in fundamentally different ways, these conditions are merely sufficient.

Theorem 1 (Comparative Precautionary Saving under RU for Income Risk) *Consider two RU decision-makers u, v with identical optimal saving amounts under the reference income $\tilde{y}_{2,l}$, and $\theta^{y_2}_u, \theta^{y_2}_v$ from* (9)*. Then, $\theta^{y_2}_u \geq \theta^{y_2}_v$ for all $\tilde{y}_{2,h}$ with $\tilde{y}_{2,h} \preceq_{n-\ell-MPSD} \tilde{y}_{2,l}$, if the conditions of Liu's Theorem 3 hold so that $\theta^{y_2}_{\psi_u} \geq \theta^{y_2}_{\psi_v}$ and, alternatively,*

(1.*i*) $\phi''_u \leq 0 \leq \phi''_v$.

(1.*ii*) *if $\phi''_u \leq \phi''_v \leq 0$, the conditions of Liu's Theorem 2 hold so that $\pi^{y_2}_{\psi_u} \geq \pi^{y_2}_{\psi_v}$.*

(1.*iii*) *if $0 \leq \phi''_u \leq \phi''_v$, the conditions of Liu's Theorem 2 hold so that $\pi^{y_2}_{\psi_u} \leq \pi^{y_2}_{\psi_v}$.*

Theorem 1 combines the preference conditions for the MEU channel, known from Liu's Theorem 3, with additional ones for the CE channel. For example, for a mean-preserving

[9] Assessing the practical implications of this requirement is important for applications, but beyond the scope of the present paper.

[10] In the other cases, the difference between $\theta^{y_2}_u$ and $\theta^{y_2}_v$ arising from the CE channel is either ambiguous at the given level of generality ($\phi''_u \leq \phi''_v \leq 0$ or $0 \leq \phi''_v \leq \phi''_u$ combined with $\pi^{y_2}_{\psi_u} \leq \pi^{y_2}_{\psi_v}$; $0 \leq \phi''_u \leq \phi''_v$ or $\phi''_v \leq \phi''_u \leq 0$ combined with $\pi^{y_2}_{\psi_u} \geq \pi^{y_2}_{\psi_v}$), or there is a stronger positive influence on $\theta^{y_2}_v$ than on $\theta^{y_2}_u$ ($\phi''_v \leq \phi''_u \leq 0$ combined with $\pi^{y_2}_{\psi_u} \leq \pi^{y_2}_{\psi_v}$; $0 \leq \phi''_v \leq \phi''_u$ combined with $\pi^{y_2}_{\psi_u} \geq \pi^{y_2}_{\psi_v}$; $\phi''_v \leq 0 \leq \phi''_u$).

spread of $\tilde{y}_2$, the fact that ψ_u is $(3/2)^{th}$-degree Ross more risk-averse than ψ_v is equivalent to decision-maker u having the stronger precautionary motive from the MEU channel. To ensure u's stronger *total* precautionary response, in addition, either of three sets of conditions must hold for the CE channel. In (1.i), the precautionary-saving contribution from the CE channel is positive for u, but negative for v. In (1.ii), both decision-makers' contributions are positive but for the same CE decrease u's is larger. Then, u has the stronger total response if ψ_u is $(2/1)^{th}$-degree Ross more risk-averse than ψ_v. In (1.iii), both contributions are negative but for the same CE decrease u's is *smaller*. In this case, ψ_v being $(2/1)^{th}$-degree Ross more risk-averse than ψ_u ensures u's stronger total response.

The last scenario reflects a situation in which ψ_v is more risk-averse than ψ_u, and v has a substantially higher EIS than u. In this setting, the "more risk-averse" decision-maker actually exhibits a *smaller* precautionary response. This departure from EU intuition is a consequence of the interacting risk and intertemporal preferences within the CE channel.

4 Increases in Return Risk

To examine the effects of deteriorations in return risk on CE, Bostian and Heinzel apply Eeckhoudt and Schlesinger's (2009) multiplicative risk premium π_ψ^R, defined by

$$\psi(CE(y_2 + s\tilde{R}_h)) = E\psi(y_2 + s(\tilde{R}_l - \pi_\psi^R)) = E\psi(y_2 + s\tilde{R}_h) \tag{13}$$

π_ψ^R measures the *proportion* of saving such that the product $s\pi_\psi^R$ is equal to the maximum amount of future consumption the decision-maker is willing to forgo to avoid the risk increase. Like $\pi_\psi^{y_2}$, π_ψ^R is positive under risk aversion, and rises under risk increases in the n-ℓ-MPSD order if and only if $\text{sgn}\left[\psi^{(k)}(.)\right] = (-1)^{k+1}$ for all $k = \ell + 1, \ldots, n$.

With a proof similar to that of Liu's Theorem 2 (which we leave to the reader), risk attitudes can be compared as follows.

Lemma 1 (Comparative Risk Aversion for Return Risk)
Suppose $n \geq 2$ and $1 \leq \ell \leq n-1$, and consider two EU functions ψ_u, ψ_v with sgn$\left[\psi_u^{(k)}(.)\right] =$ sgn$\left[\psi_v^{(k)}(.)\right] = (-1)^{k+1}$ for $k = 1$ and $\ell + 1, \ldots, n$, and identical optimal saving amounts

under the reference return $\tilde{R}_l$. Then, $\pi^R_{\psi_u} \geq \pi^R_{\psi_v}$ for all $\tilde{R}_h$ with $\tilde{R}_h \preceq_{n-\ell-MPSD} \tilde{R}_l$ and π^R_ψ from (13) *for $\psi \in \{\psi_u, \psi_v\}$ if and only if ψ_u is $(k/1)^{th}$-degree Ross more risk-averse than ψ_v for all $k = \ell + 1, \ldots, n$.*

We next extend the multiplicative premium concept to precautionary saving. For an EU function ψ, the vNM precautionary premium θ^R_ψ derives from

$$E[\psi'(y_2 + s(\tilde{R}_l - \theta^R_\psi))\tilde{R}_l] = E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] \tag{14}$$

θ^R_ψ is the proportion of saving such that the product $s\theta^R_\psi$ is the safe change in $\tilde{c}^R_2$ that generates the same effect on saving as the deterioration from $\tilde{R}_l$ to $\tilde{R}_h$.

The saving response to return risk contains a positive precautionary effect and a negative substitution effect, and so it can be positive or negative. Importantly, the sign of θ^R_ψ depends on the sign of the net saving response: θ^R_ψ is positive (negative) if and only if total saving increases (falls). To see this, note that the marginal EU premium

$$E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] - E[\psi'(y_2 + s\tilde{R}_l)\tilde{R}_l]$$

is positive (negative) if and only if the precautionary (substitution) effect dominates. Because future MEU decreases in s, saving s^{*h} under $\tilde{R}_h$ is higher (lower) than s^{*l} under $\tilde{R}_l$.[11] Future MEU on the left of (14) depends positively on θ^R_ψ, so that we have that

$$s^{*h} \gtreqless s^{*l} \quad \Leftrightarrow \quad \theta^R_\psi \gtreqless 0$$

Moreover, θ^R_ψ increases in absolute terms with the strength of the saving reaction.

Eeckhoudt and Schlesinger (2008: Proposition 2) implies that for the EU function ψ the precautionary dominates the substitution effect in response to a n-ℓ-MPSD deterioration of $\tilde{R}$, if the product of $(-1)^k$ and the k^{th} derivative of $h_\psi(R) \equiv R\psi'(y_2 + sR)$,

$$h^{(k)}_\psi(R) = \psi^{(k+1)}(y_2 + sR)s^k R + ks^{k-1}\psi^{(k)}(y_2 + sR) \tag{15}$$

[11] The marginal EU premium is evaluated at $s = s^{*l}$ (Eeckhoudt and Schlesinger 2009).

is positive for all $k = \ell+1,\dots,n$. For $\operatorname{sgn}\big[\psi^{(j)}(.)\big] = (-1)^{j+1}$ for $j = k, k+1$, the first term in this product expresses the precautionary effect and the second the substitution effect (Bostian and Heinzel). For decreasing saving, we focus in the following on the case where $(-1)^k\, h_\psi^{(k)}(R)$ is negative for *all* $k = \ell+1,\dots,n$.[12]

The statements of comparative precautionary saving under return risk differ from Liu's under income risk in two respects. First, the comparison in terms of Ross more risk aversion involves here $h_\psi^{(k)}(R)$ from (15), instead of $\psi^{(k+1)}(.)$. Second, two formulations arise, depending on whether the two saving amounts both increase or decrease.[13]

Lemma 2 (Comparative Precautionary Saving under EU for Return Risk) *Suppose $n \geq 2$ and $1 \leq \ell \leq n-1$, and consider two increasing and concave EU functions ψ_u, ψ_v with $h_\psi^{(k)}(R)$ from (15) such that $(-1)^k\, h_\psi^{(k)}(R) \geq [\,\leq\,]\ 0$ for all $k = \ell+1,\dots,n$ and $\psi \in \{\psi_u, \psi_v\}$ and identical optimal saving amounts under the reference return $\tilde{R}_l$. If both saving amounts increase [decrease] in response to a return-risk increase in the n-ℓ-MPSD order, then (i)–(iii) are equivalent:*

(i) ψ_u is Ross more risk-averse than ψ_v for $k = \ell+1,\dots,n$ in the sense that

$$\frac{h_{\psi_u}^{(k)}(R_a)}{h_{\psi_v}^{(k)}(R_a)} \geq \lambda \geq \frac{\psi_u''(c_b^R)}{\psi_v''(c_b^R)} \quad \text{for all} \;\; c_a^R, c_b^R \tag{16a}$$

(ii) There exist $\lambda > 0$ and $\eta(c^R)$ with $\eta'(c^R)$ such that

$$\psi_u'(c^R) = \lambda \psi_v'(c^R) + \eta'(c^R) \tag{16b}$$

where $\eta''(c^R) \geq 0$ and $(-1)^k\, h_\eta^{(k)}(R) \geq [\,\leq\,]\ 0$ for all c^R and $k = \ell+1,\dots,n$, with $h_\eta^{(k)}(R)$ as in (15) for $\psi = \eta$.

(iii) $\theta_{\psi_u}^R \geq \theta_{\psi_v}^R \geq 0$ $\big[\, \theta_{\psi_u}^R \leq \theta_{\psi_v}^R \leq 0 \,\big]$ for all $\tilde{R}_h$ with $\tilde{R}_h \preceq_{n-\ell-MPSD} \tilde{R}_l$ and θ_ψ^R as defined in (14) for $\psi \in \{\psi_u, \psi_v\}$.

We prove this lemma in Appendix A.

[12]For Ekern risk increases, $(-1)^n\, h_\psi^{(n)}(R) \geq 0$ is necessary and sufficient for saving to increase.

[13]Determining the larger precautionary premium is trivial if the saving responses have different signs.

Condition (16a) has an interesting interpretation, when rewritten as[14]

$$(-1)^{k+1}\left[\frac{\psi_u^{(k+1)}(c_a^R)}{\psi_u''(c_b^R)}-\frac{\psi_v^{(k+1)}(c_a^R)}{\psi_v''(c_b^R)}\right] sR \geq [\ \leq\ ]\ (-1)^k\, k\left[\frac{\psi_u^{(k)}(c_a^R)}{\psi_u''(c_b^R)}-\frac{\psi_v^{(k)}(c_a^R)}{\psi_v''(c_b^R)}\right] \tag{17}$$

If both decision-makers increase (decrease) saving, the difference of ψ_u's and ψ_v's precautionary effects is larger (smaller) than that of their substitution effects. In the context of Lemma 2, this is equivalent to ψ_u having the stronger positive (negative) precautionary response. Importantly, as applied to RU, Lemma 2 helps to compare the contributions to precautionary saving from the MEU channel.

We define the RU precautionary premium θ^R, equivalently, as the solution to

$$u'(CE(y_2+s(\tilde{R}_l-\theta_u^R)))CE'(y_2+s(\tilde{R}_l-\theta_u^R)) = u'(CE(y_2+s\tilde{R}_h))CE'(y_2+s\tilde{R}_h) \tag{18a}$$

$$\phi_u'(E\psi(y_2+s(\tilde{R}_l-\theta_u^R)))E[\psi'(y_2+s(\tilde{R}_l-\theta_u^R))\tilde{R}_l] = \phi_u'(E\psi(y_2+s\tilde{R}_h))E[\psi'(y_2+s\tilde{R}_h)\tilde{R}_h] \tag{18b}$$

The interpretation and analysis of θ_u^R and θ_ψ^R are analogous. Like θ_ψ^R, the sign of θ_u^R always mirrors the direction of the net saving response.

Inserting π_ψ^R and θ_ψ^R into (18b) links the preferences driving each risk-impact channel to the precautionary saving motive:

$$\phi_u'(E\psi(y_2+s(\tilde{R}_l-\pi_\psi^R)))E[\psi'(y_2+s(\tilde{R}_l-\theta_\psi^R))\tilde{R}_l] = \phi_u'(E\psi(y_2+s\tilde{R}_h))E[\psi'(y_2+s\tilde{R}_h)\tilde{R}_h] \tag{19}$$

As under income risk, θ_u^R rises with θ_ψ^R, but the effect of π_ψ^R is ambiguous: θ_u^R increases (decreases) with π_ψ^R if ϕ_u is concave (convex).

Theorem 2 states the sufficient conditions for observing $\theta_{\psi_u}^R \geq \theta_{\psi_v}^R \geq 0$ or $\theta_{\psi_u}^R \leq \theta_{\psi_v}^R \leq 0$ for the cases that admit such conditions.[15]

Theorem 2 (Comparative Precautionary Saving under RU for Return Risk)

[14] See Lemma 4 for the equivalence of (16a) and (17).

[15] For all combinations of curvature constellations between ϕ_u and ϕ_v and $\pi_{\psi_u}^R \geq \pi_{\psi_v}^R$ or $\pi_{\psi_u}^R \leq \pi_{\psi_v}^R$ not mentioned in Theorem 2, the order of the relative contributions to the total precautionary motives from the CE channel is either ambiguous or it is opposite to the one from the MEU channel.

Consider two RU decision-makers u, v with identical optimal saving amounts under the reference return $\tilde{R}_l$, and θ_u^R, θ_v^R from (18). Then, $\theta_u^R \geq \theta_v^R \geq 0$ $\left[\, \theta_u^R \leq \theta_v^R \leq 0 \,\right]$ for all $\tilde{R}_h$ with $\tilde{R}_h \preceq_{n-\ell-MPSD} \tilde{R}_l$, if the conditions of Lemma 2 hold so that $\theta_{\psi_u}^R \geq \theta_{\psi_v}^R \geq 0$ $\left[\, \theta_{\psi_u}^R \leq \theta_{\psi_v}^R \leq 0 \,\right]$ and, alternatively,

(2.*i*) $\phi_u'' \leq 0 \leq \phi_v''$ $[\, \phi_v'' \leq 0 \leq \phi_u'' \,]$.

(2.*ii*) *if* $\phi_u'' \leq \phi_v'' \leq 0$ $[\, 0 \leq \phi_v'' \leq \phi_u'' \,]$*, the conditions of Lemma 1 hold so that* $\pi_{\psi_u}^R \geq \pi_{\psi_v}^R$.

(2.*iii*) *if* $0 \leq \phi_u'' \leq \phi_v''$ $[\, \phi_v'' \leq \phi_u'' \leq 0 \,]$*, the conditions of Lemma 1 hold so that* $\pi_{\psi_u}^R \leq \pi_{\psi_v}^R$.

To illustrate the theorem, consider a mean-preserving spread of $\tilde{R}$ that makes both decision-makers save less. Under Theorem 2, the preference conditions for the MEU channel come from Lemma 2. In (2.i), the fact that ψ_u is Ross more risk-averse than ψ_v in the sense that $\frac{h''_{\psi_u}(R_a)}{h''_{\psi_v}(R_a)} \geq \lambda \geq \frac{\psi_u''(c_b^R)}{\psi_v''(c_b^R)}$ is equivalent to decision-maker u having the stronger negative precautionary saving reaction. The additional conditions for the CE channel in (2.ii) and (2.iii) rely on Lemma 1. In (2.ii), the precautionary-saving contributions from the CE channel are negative for u and v but for the same CE decrease u's is larger. Then, u has the stronger total response if ψ_u is also $(2/1)^{th}$-degree Ross more risk-averse than ψ_v. In (2.iii), the contributions from the CE channel are positive for both but for the same CE decrease u's is *smaller*. In this case, u has the stronger total response if, in addition, ψ_v is $(2/1)^{th}$-degree Ross more risk-averse than ψ_u.

As compared to income risk, the example illustrates that the same type of risk increase (mean-preserving spread) on a different type of risk ($\tilde{R}$ instead of $\tilde{y}_2$) activates different preferences only in the MEU channel. The CE channels are supported by the same kind of preferences. The RU precautionary premia from (9) and (18) help to measure the associated precautionary saving motives conveniently in single quantities. But their conceptual difference makes the two premia incomparable: the first takes consumption units, while the second has a percentage scale. Also, these premia do not detail the intensities of the different preferences sustaining a precautionary choice. We turn next to alternative characterizations of the involved preferences using preference coefficients.

5 Representation by Preference Coefficients

To link Ross more risk aversion to preference coefficients, Liu and Meyer (2013a) develop a measure similar to the Arrow-Pratt coefficient of absolute risk aversion

$$A_f(x) \equiv -\frac{f''(x)}{f'(x)}$$

The only difference is that Liu and Meyer's measure evaluates f' at a fixed value x_0 within a bounded interval $[a, b]$. Assuming the monotonicity of f' and negativity of f'' over this interval, they show that the concavity measure

$$C_f(x; a) \equiv -\frac{f''(x)}{f'(a)}$$

equivalently represents Ross more risk aversion.[16]

To extend this measure to higher-order preferences, we apply Liu and Meyer's (2013b) measure of local $(n/m)^{th}$-degree absolute risk aversion, defined for an n-times differentiable function f with a positive (negative) m^{th} derivative for m odd (even),

$$A_{(n/m)_f}(x) = \frac{(-1)^{n-1} f^{(n)}(x)}{(-1)^{m-1} f^{(m)}(x)}$$

Definition 3 (Generalized Concavity Measure) *For an n-times differentiable function $f(x)$ with $sgn\big[f^{(m)}(x)\big] = (-1)^{m+1}$ and $n > m \geq 1$ defined on $[a, b] \subseteq \mathbb{R}_0^+$, the generalized concavity measure of $(n/m)^{th}$-degree risk aversion is*

$$C_{(n/m)_f}(x; a) = (-1)^{n-m} \frac{f^{(n)}(x)}{f^{(m)}(a)} \tag{20}$$

We first amend $(n/m)^{th}$-degree Ross more risk aversion as in Definition 1 by a representation with preference coefficients.

[16] The sufficiency part of this equivalence follows because the condition for Ross more risk aversion, $\frac{u''(x_a)}{v''(x_a)} \geq \lambda \geq \frac{u'(x_b)}{v'(x_b)}$ for all $x_a, x_b \in [a, b]$, implies for $x_b = a$ that $C_u(x; a) \geq C_v(x; a)$ for all $x_a \in [a, b]$. Necessity follows by integrating the latter inequality on both sides from a to x_b, yielding $\frac{u'(x_b)}{u'(a)} \leq \frac{v'(x_b)}{v'(a)}$ for all $x_b \in [a, b]$. These two inequalities, combined with $\lambda = \frac{u'(a)}{v'(a)}$, establish the equivalence.

Lemma 3 (Comparative $(n/m)^{th}$-Degree Risk Aversion using Coefficients) *Let ψ_u be more $((m+1)/m)^{th}$-degree risk-averse than ψ_v, i.e., $-\frac{\psi_u^{(m+1)}(x)}{\psi_u^{(m)}(x)} \geq -\frac{\psi_v^{(m+1)}(x)}{\psi_v^{(m)}(x)}$ for all $x \in [a,b]$. Then, condition (8) is equivalent to*

$$C_{(n/m)_{\psi_u}}(x;a) \geq C_{(n/m)_{\psi_v}}(x;a) \quad \textit{for all} \quad x \in [a,b] \tag{21}$$

with $C_{(n/m)_f}$ as in (20) for $f \in \{\psi_u, \psi_v\}$.

We prove this lemma in Appendix B.

For $m = 1$, Lemma 3 provides a complementary characterization of n^{th}-degree Ross more risk aversion (Li 2009, Denuit and Eeckhoudt 2010a) using generalized concavity measures of $(n/1)^{th}$-degree risk aversion. Additionally, when applied for $k = \ell+1, \ldots, n$, it gives representations of Liu's Theorem 2 and our Lemma 1 involving $C_{(k/1)}$ coefficients. For $m = 2$, the lemma provides the conditions to compare the strengths of EU precautionary saving motives under income risk (Liu: Theorem 3) with $C_{((k+1)/2)}$ coefficients. Because higher-order risk changes entail positive variance by definition, ψ_u must also be globally more Arrow-Pratt risk-averse (Kimball-prudent) than ψ_v for $m = 1$ ($m = 2$).

Thanks to Lemma 3, we can thus compare the risk preferences active in the CE channels and, for income risk, the MEU channel using coefficients. Lemma 4 considers the MEU channel under return risk.

Lemma 4 (Coefficient Representation of Lemma 2) *Let ψ_u be more Kimball-prudent than ψ_v, i.e., $-\frac{\psi_u'''(c^R)}{\psi_u''(c^R)} \geq -\frac{\psi_v'''(c^R)}{\psi_v''(c^R)}$ for all $c^R \in [a,b]$, and both decision-makers increase [decrease] saving in response to a return-risk increase in the n-ℓ-MPSD order. Then, condition (16a) for $c_a^R, c_b^R \in [a,b]$ is equivalent to*

$$\left[C_{((k+1)/2)_{\psi_u}}(c^R;a) - C_{((k+1)/2)_{\psi_v}}(c^R;a)\right] sR \geq [\ \leq\ ]\ k \left[C_{(k/2)_{\psi_u}}(c^R;a) - C_{(k/2)_{\psi_v}}(c^R;a)\right] \tag{22}$$

for all $c^R \in [a,b]$ and $k = \ell+1, \ldots, n$, with $C_{(j/2)_f}$ for $j \in \{k, k+1\}$ as in (20) and $C_{(2/2)_f}(c^R;a) \equiv \frac{f''(c^R)}{f''(a)}$, for $f \in \{\psi_u, \psi_v\}$.

The proof of Lemma 4 is similar to that of Lemma 3. Crucial to recognize is that

$\frac{h^{(k)}_{\psi_u}(R)}{h^{(k)}_{\psi_v}(R)} \geq \frac{\psi''_u(a)}{\psi''_v(a)}$ (as implied by (16a)) is, with $h^{(k)}_{\psi}(R)$ from (15) for $\psi \in \{\psi_u, \psi_v\}$, equivalent to a version of (17) with $(c^R_a, c^R_b) = (c^R, a)$. With this substitution, the $C_{((k+1)/2)}$ coefficients from (20) control the precautionary effects and the $C_{(k/2)}$ coefficients control the substitution effects in this inequality. Inserting these coefficients yields (22). Interestingly, while the precautionary effects depend on the same coefficients as under income risk, the coefficients controlling the substitution effects differ from those the CE channels involve ($C_{(k/2)}$ instead of $C_{(k/1)}$).

Theorem 3 summarizes the results for RU. Along with the curvature condition (6) for ϕ, it provides the conditions to predict from preference coefficients when one RU decision-maker has a stronger precautionary saving motive than another. Its statements detail the ways in which the different preferences are involved in precautionary saving.

Theorem 3 (Comparative Precautionary Saving under RU using Coefficients) *For two RU decision-makers u, v, let ψ_u be more prudent than ψ_v, i.e., $-\frac{\psi'''_u(c)}{\psi''_u(c)} \geq -\frac{\psi'''_v(c)}{\psi''_v(c)}$ for all $c \in [a, b]$. Then, for Theorem 1, with $c = c^{y_2}$, $\theta^{y_2}_u \geq \theta^{y_2}_v$ follows in $(1.i)$ also if*

$$C_{((k+1)/2)_{\psi_u}}(c^{y_2}; a) \geq C_{((k+1)/2)_{\psi_v}}(c^{y_2}; a) \quad \textit{for all} \quad c^{y_2} \in [a, b]$$

For Theorem 2, with $c = c_R$, $\theta^R_u \geq \theta^R_v \geq 0$ $\left[\, \theta^R_u \leq \theta^R_v \leq 0 \,\right]$ follows in $(2.i)$ also if

$$\left[C_{((k+1)/2)_{\psi_u}}(c^R; a) - C_{((k+1)/2)_{\psi_v}}(c^R; a)\right] sR \geq [\,\leq\,]\; k \left[C_{(k/2)_{\psi_u}}(c^R; a) - C_{(k/2)_{\psi_v}}(c^R; a)\right]$$

for all $c^R \in [a, b]$ and $k = \ell + 1, \ldots, n$, with $C_{(j/2)_f}$ for $j \in \{k, k+1\}$ as in Lemma 4.

$(1.ii)$ and $(2.ii)$ additionally require that ψ_u be more Arrow-Pratt risk-averse than ψ_v, i.e., $-\frac{\psi''_u(c)}{\psi'_u(c)} \geq -\frac{\psi''_v(c)}{\psi'_v(c)}$, and $C_{(k/1)_{\psi_u}}(c; a) \geq C_{(k/1)_{\psi_v}}(c; a)$ for all $c \in [a, b]$ and $k = \ell + 1, \ldots, n$.

$(1.iii)$ and $(2.iii)$ additionally require the reverse relations regarding the risk aversion of ψ_u and ψ_v as compared to $(1.ii)$ and $(2.ii)$, respectively.

Theorem 3 illustrates the increased complexity of comparing precautionary saving motives under RU. Thus, for income risk under EU it is enough to inspect $((k+1)/2)^{th}$-degree risk aversion (Lemma 3). For example, an increase in downside risk on $\tilde{y}_2$ (a 3-2-MPSD shift) makes ψ_u exhibit a stronger precautionary motive than ψ_v if ψ_u is globally more prudent than ψ_v and their generalized concavity measures of temperance $C_{(4/2)_\psi}$ compare as $\frac{\psi_u''''(c^{y_2})}{\psi_u''(a)} \geq \frac{\psi_v''''(c^{y_2})}{\psi_v''(a)}$. Comparing two RU decision-makers entails, in addition to this same set of criteria for the MEU channel, requirements specific to the CE channel. Only if second-order risk and intertemporal preferences are globally such that $ARIS_u > ARA_{\psi_u}$ and $ARIS_v < ARA_{\psi_v}$, so that $\phi_u'' \leq 0 \leq \phi_v''$, the conditions for the MEU channel are enough to determine whether $\theta_u^{y_2} \geq \theta_v^{y_2}$. If both functions exhibit $ARIS$ greater (less) than ARA_ψ, and these coefficients entail $\phi_u'' \leq \phi_v'' \leq 0$ ($0 \leq \phi_u'' \leq \phi_v''$), we must additionally compare their Arrow-Pratt risk aversion and their generalized concavity measures of downside risk aversion $C_{(3/1)_\psi}(c^{y_2}; a) = \frac{\psi'''(c^{y_2})}{\psi'(a)}$. Similar comparisons for an increase in downside risk on $\tilde{R}$ involve, in addition, in (22) the $C_{(3/2)_\psi}$ coefficients.

Like Theorems 1 and 2, Theorem 3 provides no representation when the CE channel renders ambiguous or counter-directional effects.[17] These inconclusive assessments occur at a rather high level of theoretical generality. Exploring their relevance in real settings, which could uncover additional regularities in these cases, is a future task.

Lemmas 3 and 4 and Theorem 3 show the flexible applicability of generalized concavity measures for different kinds of preference comparisons, in static and dynamic contexts, for additive and multiplicative risks, and under EU and RU. Unlike the premia, these coefficients span income risk and return risk. Coefficients are widely used in applications. This approach may thus provide a lower bar for assessing precautionary motives.

6 Relation to Existing Literature

Our investigation merges three streams of literature. In the first, Eeckhoudt and Schlesinger (2008) derive the conditions on utility functions for risk-induced higher saving. Building on their results, Liu generalizes Kimball's precautionary-premium analysis to higher-order income risks. These studies focus on EU and do not consider preference coefficients.

[17] For higher-order risks, Theorem 3 technically has stronger conditions than Theorems 1 and 2 because of the required additional comparison of preferences towards second-order risk.

In another stream, Kimball and Weil (2009) extend Kimball's work on precautionary saving to RU. Their decision framework, which we adopt here, is rooted in Kreps and Porteus (1978) and Selden (1978), and was popularized by Epstein and Zin (1989, 1991) and Weil (1990). RU has become a standard modeling approach in settings where risk aversion and intertemporal substitution matter simultaneously. Few studies under RU address precaution. To our knowledge, KW is the only constructive contribution to the measurement and comparison of precautionary saving motives under RU.

The final stream discusses the proper assessment of risk preferences depending on the risk environment. Both KW and Kimball formulate their precautionary premia and prudence coefficients only for additive risks on nonrandom future income, akin to the scope of the Arrow-Pratt measures of risk aversion. Unfortunately, these measures are not well suited for even slightly more complicated settings, including changes in risk (Kihlstrom et al. 1981, Ross 1981), multiplicative risk (Capéraà and Eeckhoudt 1975, Briys et al. 1989), and higher-order risk (Chiu 2005, Denuit and Eeckhoudt 2010b).

Under return risk, Drèze and Modigliani (1972) for EU and Langlais (1995) for RU suggest premium measures that suffer from issues similar to the ones raised against the application of the Arrow-Pratt premium with multiplicative risk. (Arrow-Pratt premia need not be positive under risk aversion, nor increase with a mean-preserving spread, in this case.) Briys et al.'s additive premium avoids these problems by accounting for the optimal choices in the low- and high-risk states. Eeckhoudt and Schlesinger's (2009) multiplicative premium concept, which we adopt, provides a more convenient solution.

Ross's stronger conditions admit comparisons of risk aversion for increases in additive risk. Higher-order extensions of this approach (e.g., Liu and Meyer 2013b) can be applied to multiplicative risks using multiplicative premia, as we show. Thanks to the central role of risk preferences under RU, the latter extensions of Ross's approach under EU help to formulate the conditions for comparative precautionary saving under RU.

7 Conclusion

We extend the analysis of comparative precautionary saving to higher-order risk effects under RU and return risk, covering (to the possible extent) the broad set of risks in Eeckhoudt and Schlesinger (2008). Under RU, these comparative statements always involve risk and intertemporal preferences, spawning additional behavioral tradeoffs compared to EU. For return risk, we define a multiplicative precautionary premium starting from Eeckhoudt and Schlesinger (2009). But its sign ambivalence and conceptual difference with the precautionary premium under income risk may impair the merit of such premia in applications. Our alternative representation with preference coefficients gauges more finely the preferences activated under income risk and return risk. We hope that our results can help to better understand saving behavior in various risky environments.

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Appendix

A Proof of Lemma 2

Consider first the case where both saving amounts increase in response to a given return-risk increase, so that $\theta^R_{\psi_u}, \theta^R_{\psi_v} \geq 0$.

(i) $\Rightarrow$ (ii) is analogous to Liu (2014).

(ii) $\Rightarrow$ (iii). With (14) for $\psi \in \{\psi_u, \psi_v\}$ and the concavity of ψ_u,

$$\theta^R_{\psi_u} \geq \theta^R_{\psi_v} \;\Leftrightarrow\; E[\psi'_u(y_2 + s(\tilde{R}_l - \theta^R_{\psi_u}))\tilde{R}_l] \geq E[\psi'_u(y_2 + s(\tilde{R}_l - \theta^R_{\psi_v}))\tilde{R}_l] \tag{23}$$

Given $\eta(c^R)$ with $\eta'(c^R)$ such that (16b) holds, $\eta''(c^R) \geq 0$, and $(-1)^k h^{(k)}_\eta(R) \geq 0$ for all c^R and $k = \ell+1, \ldots, n$, (23) holds because

$$\begin{aligned}
E[\psi'_u(y_2 + s(\tilde{R}_l - \theta^R_{\psi_u}))\tilde{R}_l] &= E[\psi'_u(y_2 + s\tilde{R}_h)\tilde{R}_h] \\
&= \lambda E[\psi'_v(y_2 + s\tilde{R}_h)\tilde{R}_h] + E[\eta'(y_2 + s\tilde{R}_h)\tilde{R}_h] \\
&\geq \lambda E[\psi'_v(y_2 + s\tilde{R}_h)\tilde{R}_h] + E[\eta'(y_2 + s\tilde{R}_l)\tilde{R}_l] \\
&= \lambda E[\psi'_v(y_2 + s(\tilde{R}_l - \theta^R_{\psi_v}))\tilde{R}_l] + E[\eta'(y_2 + s\tilde{R}_l)\tilde{R}_l] \\
&\geq \lambda E[\psi'_v(y_2 + s(\tilde{R}_l - \theta^R_{\psi_v}))\tilde{R}_l] + E[\eta'(y_2 + s(\tilde{R}_l - \theta^R_{\psi_v}))\tilde{R}_l] \\
&= E[\psi'_u(y_2 + s(\tilde{R}_l - \theta^R_{\psi_v}))\tilde{R}_l],
\end{aligned}$$

The first line is (14) for $\psi = \psi_u$; the second line applies (16b); the first inequality follows from the NSD equivalence in Eeckhoudt and Schlesinger (2008) for $f = \eta'$ applied to the case of return-risk increases in the n-ℓ-MPSD order; the fourth line applies (14) for $\psi = \psi_v$; the second inequality follows from $\eta''(c^R) \geq 0$ and $\theta^R_{\psi_v} \geq 0$; and the last line applies (16b) again.

(iii) $\Rightarrow$ (i). $\theta^R_{\psi_u} \geq \theta^R_{\psi_v} \geq 0$ for all $\tilde{R}_h$ such that $\tilde{R}_h \preceq_{n-\ell-MPSD} \tilde{R}_l$ and θ^R_ψ defined as in (14) for $\psi \in \{\psi_u, \psi_v\}$ implies that $\theta^R_{\psi_u} \geq \theta^R_{\psi_v} \geq 0$ for all $\tilde{R}_h$ such that $\tilde{R}_h$ is a k^{th}-degree Ekern (1980) risk increase over $\tilde{R}_l$ for all $k = \ell+1, \ldots, n$. Based on an argument analogous to Denuit and Eeckhoudt (2010a), this implies that ψ_u is Ross more risk averse than ψ_v in the sense of (16a) for all $k = \ell + 1, \ldots, n$.

When both saving amounts instead decrease, so that $\theta^R_{\psi_u}, \theta^R_{\psi_v} \leq 0$, (i) $\Rightarrow$ (ii) follows as before, only that $(-1)^k h^{(k)}_{\psi}(R) \leq 0$ for $\psi \in \{\psi_u, \psi_v\}$ so that $(-1)^k\, h^{(k)}_{\eta}(R) \leq 0$, for $k = \ell + 1, \ldots, n$. In (ii) $\Rightarrow$ (iii), only the directions of the two inequalities are reversed, because of $(-1)^k\, h^{(k)}_{\eta}(R) \leq 0$ and $\theta^R_{\psi_v} \leq 0$, respectively. For (iii) $\Rightarrow$ (i), the proof starting from $\theta^R_{\psi_u} \leq \theta^R_{\psi_v} \leq 0$ is analogous. ■

B Proof of Lemma 3

Sufficiency. Given (8), there exists for all $m < n$ and all $x_a, x_b \in [a, b]$ a $\lambda > 0$ such that

$$\frac{\psi^{(n)}_u(x_a)}{\psi^{(n)}_v(x_a)} \geq \lambda \geq \frac{\psi^{(m)}_u(x_b)}{\psi^{(m)}_v(x_b)}\,. \tag{24}$$

Let $x_b = a$. Then, for all $x \in [a, b]$,

$$\frac{\psi^{(n)}_u(x)}{\psi^{(n)}_v(x)} \geq \frac{\psi^{(m)}_u(a)}{\psi^{(m)}_v(a)} \quad \Leftrightarrow \quad (-1)^{n-m} \frac{\psi^{(n)}_u(x)}{\psi^{(m)}_u(a)} \geq (-1)^{n-m} \frac{\psi^{(n)}_v(x)}{\psi^{(m)}_v(a)}\,.$$

Necessity. Given (20), so that, for $x = x_a$,

$$(-1)^{n-m} \frac{\psi^{(n)}_u(x_a)}{\psi^{(m)}_u(a)} \geq (-1)^{n-m} \frac{\psi^{(n)}_v(x_a)}{\psi^{(m)}_v(a)} \quad \Leftrightarrow \quad \frac{\psi^{(n)}_u(x_a)}{\psi^{(n)}_v(x_a)} \geq \frac{\psi^{(m)}_u(a)}{\psi^{(m)}_v(a)}\,.$$

The first inequality in (24) arises by setting $\lambda \equiv \frac{\psi^{(m)}_u(a)}{\psi^{(m)}_v(a)}$. The second holds because $\frac{\psi^{(m)}_u(a)}{\psi^{(m)}_v(a)}$ decreases with its argument if, for all $x \in [a, b]$, $-\frac{\psi^{(m+1)}_u(x)}{\psi^{(m)}_u(x)} \geq -\frac{\psi^{(m+1)}_v(x)}{\psi^{(m)}_v(x)}$. ■